1 (a) The function s represents the grid and is given by

$$s(u_1,u_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(u_1 - n_1\Delta_1, u_2 - n_2\Delta_2)$$

[10%]

(b) We can write s as a Fourier series:

$$s(u_1,u_2) = \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} c(p_1,p_2)\, e^{j(p_1\Omega_1 u_1 + p_2\Omega_2 u_2)}$$

where $\Omega_1 = \frac{2\pi}{\Delta_1}$ and $\Omega_2 = \frac{2\pi}{\Delta_2}$.

We can then find the Fourier coefficients c in the usual way:

$$c(p_1,p_2) = \frac{1}{\Delta_1\Delta_2} \int_{-\frac{\Delta_2}{2}}^{\frac{\Delta_2}{2}} \int_{-\frac{\Delta_1}{2}}^{\frac{\Delta_1}{2}} s(u_1,u_2)\, e^{-j(p_1\Omega_1 u_1 + p_2\Omega_2 u_2)}\, du_1\, du_2$$

$$= \frac{1}{\Delta_1\Delta_2} \int_{-\frac{\Delta_2}{2}}^{\frac{\Delta_2}{2}} \int_{-\frac{\Delta_1}{2}}^{\frac{\Delta_1}{2}} \left[\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(u_1 - n_1\Delta_1, u_2 - n_2\Delta_2)\right] \times e^{-j(p_1\Omega_1 u_1 + p_2\Omega_2 u_2)}\, du_1\, du_2$$

$$\Longrightarrow c(p_1,p_2) = \frac{1}{\Delta_1\Delta_2} \quad \text{for all} \quad p_1,p_2$$

The sampled image may then be expressed as:

$$g_s(u_1,u_2) = g(u_1,u_2)\frac{1}{\Delta_1\Delta_2} \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} e^{j(p_1\Omega_1 u_1 + p_2\Omega_2 u_2)}$$

Using the frequency shift or spatial modulation theorem to take the Fourier transform

$$g(u_1,u_2)e^{j(p_1\Omega_1 u_1 + p_2\Omega_2 u_2)} \Leftrightarrow G(\omega_1 - \Omega_1 p_1, \omega_2 - \Omega_2 p_2)$$

gives:

$$\boxed{G_s(\omega_1,\omega_2) = \frac{1}{\Delta_1\Delta_2} \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} G(\omega_1 - p_1\Omega_1, \omega_2 - p_2\Omega_2)}$$

(cont.

It can therefore be seen that the Fourier transform or spectrum of the sampled 2d signal is the periodic repetition of the spectrum of the unsampled 2d signal – precisely analogous to the 1d case. It is therefore clear that for a bandlimited 2d signal, we must sample at more than twice the largest frequencies in the signal to keep these copies of the FT separate. Hence

$$\frac{2\pi}{\Delta_1} > 2\Omega_{B1} \quad \frac{2\pi}{\Delta_2} > 2\Omega_{B2}$$

These are the Nyquist frequencies, and if we sample below these we observe artefacts which we call aliasing. [20%]

(c) Now suppose we sample on the diamond grid of Fig.1.

It is not hard to see that we can express the sampling grip s_d as the sum of two rectangular grids where

$$s_1(u_1,u_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta[u_1 - 2n_1\Delta_1, u_2 - 2n_2\Delta_2]$$

$$s_2(u_1,u_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta[u_1 - (2n_1+1)\Delta_1, u_2 - (2n_2+1)\Delta_2]$$

Since s_1 and s_2 are periodic (with periods $2\Delta_1$ and $2\Delta_2$) we can represent them as 2d Fourier series.

$$s_1(u_1,u_2) = \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} c_1(p_1,p_2)\, e^{j(p_1\frac{\Omega_1}{2}u_1 + p_2\frac{\Omega_2}{2}u_2)}$$

$$s_2(u_1,u_2) = \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} c_2(p_1,p_2)\, e^{j(p_1\frac{\Omega_1}{2}u_1 + p_2\frac{\Omega_2}{2}u_2)}$$

$$\text{where } \Omega_1 = \frac{2\pi}{\Delta_1} \quad \text{and} \quad \Omega_2 = \frac{2\pi}{\Delta_2}$$

Note, we want our expressions in terms of Ω_1 and Ω_2, the frequencies of the original grid. Thus the sampled function can be written as

$$g_s(u_1,u_2) = g(u_1,u_2)[s_1(u_1,u_2) + s_2(u_1,u_2)]$$

$$= \sum_{p_1=-\infty}^{\infty} \sum_{p_2=-\infty}^{\infty} [c_1(p_1,p_2) + c_2(p_1,p_2)]\, g(u_1,u_2)\, e^{j(p_1\frac{\Omega_1}{2}u_1 + p_2\frac{\Omega_2}{2}u_2)}$$

(TURN OVER for continuation of SOLUTION 1

Take the 2-D Fourier transform using the frequency shift theorem to give:

$$G_s(\omega_1,\omega_2) = \sum_{p_1=-\infty}^{\infty}\sum_{p_2=-\infty}^{\infty} [c_1(p_1,p_2)+c_2(p_1,p_2)]\, G(\omega_1 - p_1\frac{\Omega_1}{2}, \omega_2 - p_2\frac{\Omega_2}{2})$$

By the usual process we can determine the Fourier coefficients

$$c_1(p_1,p_2) = \frac{1}{4\Delta_1\Delta_2}\int_{-\Delta_2}^{\Delta_2}\int_{-\Delta_1}^{\Delta_1} s_1(u_1,u_2)\, e^{-j(p_1\frac{\Omega_1}{2}u_1+p_2\frac{\Omega_2}{2}u_2)}\, du_1\, du_2$$

$$c_2(p_1,p_2) = \frac{1}{4\Delta_1\Delta_2}\int_{-\Delta_2}^{\Delta_2}\int_{-\Delta_1}^{\Delta_1} s_2(u_1,u_2)\, e^{-j(p_1\frac{\Omega_1}{2}u_1+p_2\frac{\Omega_2}{2}u_2)}\, du_1\, du_2$$

Clearly the factor $\frac{1}{4\Delta_1\Delta_2}$ comes from $\frac{1}{T_1T_2}$ with $T_i = 2\Delta_i$.
Substitute for $s_1(u_1,u_2)$ and interchange integral and summation operations to give:

$$c_1(p_1,p_2) = \frac{1}{4\Delta_1\Delta_2}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\int_{-\Delta_2}^{\Delta_2}\int_{-\Delta_1}^{\Delta_1} \delta[u_1 - 2n_1\Delta_1, u_2 - 2n_2\Delta_2]$$
$$\times\, e^{-j(p_1\frac{\Omega_1}{2}u_1+p_2\frac{\Omega_2}{2}u_2)}\, du_1\, du_2$$

Since the only contribution to the integral will come when $n_1 = n_2 = 0$, we have

$$\therefore\ c_1(p_1,p_2) = \frac{1}{4\Delta_1\Delta_2}$$

Similarly, substituting for $s_2(u_1,u_2)$ and interchanging integral and summation operations gives

$$c_2(p_1,p_2) = \frac{1}{4\Delta_1\Delta_2}\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty}\int_{0}^{2\Delta_2}\int_{0}^{2\Delta_1} \delta[u_1 - (2n_1+1)\Delta_1, u_2 - (2n_2+1)\Delta_2]$$
$$\times\, e^{-j(p_1\frac{\Omega_1}{2}u_1+p_2\frac{\Omega_2}{2}u_2)}\, du_1\, du_2$$

note here that we change our limits to take the period between $[0,2\Delta_1]$ and $[0,2\Delta_2]$ (recall that we can take any interval covering a whole period), so that the only contributions to the integral now come from $n_1 = 0$ and $n_2 = 0$ giving $u_1 = \Delta_1$ and $u_2 = \Delta_2$. Since $(p_i\Omega_i\Delta_i)/2 = p_i\pi$ we have

$$c_2(p_1,p_2) = \frac{1}{4\Delta_1\Delta_2}e^{-j(p_1+p_2)\pi}$$

Substituting for $c_1(p_1,p_2)$ and $c_2(p_1,p_2)$ in the equation for the sampled signal spectrum gives:

(cont.

$$G_s(\omega_1,\omega_2) = \frac{1}{4\Delta_1\Delta_2}\sum_{p_1=-\infty}^{\infty}\sum_{p_2=-\infty}^{\infty}[1+e^{-j(p_1+p_2)\pi}]G(\omega_1 - p_1\frac{\Omega_1}{2}, \omega_2 - p_2\frac{\Omega_2}{2})$$

Since $1+e^{-j(p_1+p_2)\pi} = \begin{cases} 0, & p_1+p_2 = \text{odd} \\ 2, & p_1+p_2 = \text{even} \end{cases}$

$$\boxed{G_s(\omega_1,\omega_2) = \frac{1}{2\Delta_1\Delta_2}\sum_{p_1=-\infty}^{\infty}\sum_{p_2=-\infty}^{\infty} G(\omega_1 - p_1\frac{\Omega_1}{2}, \omega_2 - p_2\frac{\Omega_2}{2}) \quad \text{for } p_1+p_2 \text{ even}}$$

[20%]

It can be seen that the spectrum of the sampled signal is the periodic repetition of the unsampled signal spectrum, where the unsampled spectrum repeats itself at (even,even) and (odd,odd) intervals of $\Omega_1/2$ and $\Omega_2/2$ as shown in figure 1.

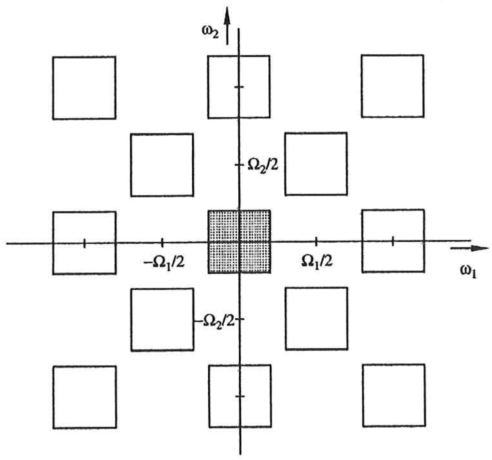


Fig. 1

10%
[~~5%~~]

(d) We know that the ideal impulse response, $h(n_1\Delta_1, n_2\Delta_2)$ is given by

(TURN OVER for continuation of SOLUTION 1

$$h(n_1\Delta_1, n_2\Delta_2) = \frac{\Delta_1\Delta_2}{(2\pi)^2}\int_{-\pi/\Delta_2}^{\pi/\Delta_2}\int_{-\pi/\Delta_1}^{\pi/\Delta_1} H(\omega_1,\omega_2)e^{j(\omega_1 n_1\Delta_1+\omega_2 n_2\Delta_2)}d\omega_1 d\omega_2$$

$$= \frac{\Delta_1\Delta_2}{(2\pi)^2}\int\int_R e^{j(\omega_1 n_1\Delta_1+\omega_2 n_2\Delta_2)}d\omega_1 d\omega_2$$

where R is the region where $H = 1$. Noting that the lines of the rotated square have equations $\omega_2 = \omega_1 \pm \Omega_s$ and $\omega_2 = -\omega_1 \pm \Omega_s$, it seems sensible to change to the following variables

$$\omega_1{}' = \omega_1 + \omega_2 \quad \text{and} \quad \omega_2{}' = \omega_1 - \omega_2$$

so that the diagonal lines in R become lines of constant ω_1' or ω_2'. Thus the integral becomes

$$\frac{\Delta_1\Delta_2}{(2\pi)^2}\int_{-\sqrt{2}\Omega_c}^{\sqrt{2}\Omega_c}\int_{-\sqrt{2}\Omega_c}^{\sqrt{2}\Omega_c} e^{j\frac{\omega_1'}{2}(n_1\Delta_1+n_2\Delta_2)}e^{j\frac{\omega_2'}{2}(n_1\Delta_1-n_2\Delta_2)}|J|d\omega_1' d\omega_2'$$

where $|J|$ is the magnitude of the jacobian of the transformation from (ω_1,ω_2) to (ω_1',ω_2') ; $|J| = \left|\frac{\partial\omega_1}{\partial\omega_1'}\frac{\partial\omega_2}{\partial\omega_2'} - \frac{\partial\omega_1}{\partial\omega_2'}\frac{\partial\omega_2}{\partial\omega_1'}\right| = \frac{1}{2}$ and $\Omega_s = \sqrt{2}\Omega_c$, and the side of the square is $2\Omega_c$.
Thus, the integral can be evaluated as

$$\frac{\Delta_1\Delta_2}{2(2\pi)^2}\left[\frac{e^{j\frac{\omega_1'}{2}(n_1\Delta_1+n_2\Delta_2)}}{\frac{j}{2}(n_1\Delta_1+n_2\Delta_2)}\right]_{-\sqrt{2}\Omega_c}^{\sqrt{2}\Omega_c}\left[\frac{e^{j\frac{\omega_2'}{2}(n_1\Delta_1-n_2\Delta_2)}}{\frac{j}{2}(n_1\Delta_1-n_2\Delta_2)}\right]_{-\sqrt{2}\Omega_c}^{\sqrt{2}\Omega_c}$$

Giving

$$\frac{\Delta_1\Delta_2}{(\pi)^2}\Omega_c{}^2 \operatorname{sinc}\frac{(n_1\Delta_1+n_2\Delta_2)\Omega_c}{\sqrt{2}} \operatorname{sinc}\frac{(n_1\Delta_1-n_2\Delta_2)\Omega_c}{\sqrt{2}}$$

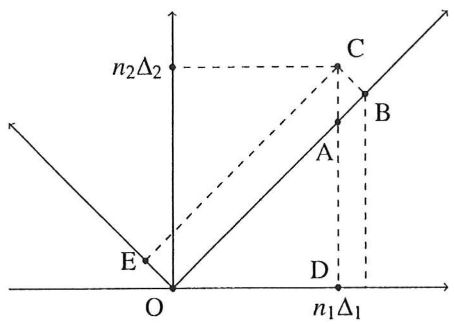
as required (replacing Ω_s with $\sqrt{2}\Omega_c$). [25%] [~~30%~~]

(e) The standard result for a square low-pass filter of side length 2Ω is

$$h(n_1\Delta_1, n_2\Delta_2) = \frac{\Delta_1\Delta_2}{(\pi)^2}\Omega^2 \operatorname{sinc}(n_1\Delta_1\Omega) \operatorname{sinc}(n_2\Delta_2\Omega) \tag{1}$$

If we change coordinates as described above in part (d), then we effectively get a square low-pass filter with side $2\Omega_c$ – however, the result above is for rectangular sampling, and if we rotate coordinates we will not have rectangular sampling.

(cont.



From the sketch above we see that a point $(n_1\Delta_1, n_2\Delta_2)$ in the rectangular axes will go to a point whose coordinates are described by B and E above. So, since $OA = \sqrt{2}n_1\Delta_1$, $AD = n_1\Delta_1$, $AC = n_2\Delta_2 - n_1\Delta_1$, $AB = BC = \frac{1}{\sqrt{2}}AC$. Therefore the new coordinates are

$$\left(\frac{(n_1\Delta_1 + n_2\Delta_2)}{\sqrt{2}}, \frac{(n_2\Delta_2 - n_1\Delta_1)}{\sqrt{2}}\right)$$

in the set of perpendicular axes at 45 degrees. Hence, replacing the values in 1, we see that we do indeed obtain the expression in part (d). [15%]

2 (a) (i) A histogram plot of frequency of occurrence of grey levels in an image against grey level will tell us how much the available grey levels are used. An intuitively appealing idea would be to apply a transformation or mapping to the image pixels in such a way that the probability of occurrence of the various grey levels should be constant i.e. all grey levels are equiprobable which would correspond to a constant amplitude histogram. This process is called *histogram equalisation.*

Histogram equalisation is often useful in bringing out detail in images which make poor use of the available grey levels – this may occur due to poor illumination of the scene or non-linearity in the imaging system. [10%]

(ii) The histogram of the image in Figure 3 (of the paper) is shown below. We can see that the grey levels used are concentrated around the middle of the range 1-18 and low and high levels are unused. [10%]

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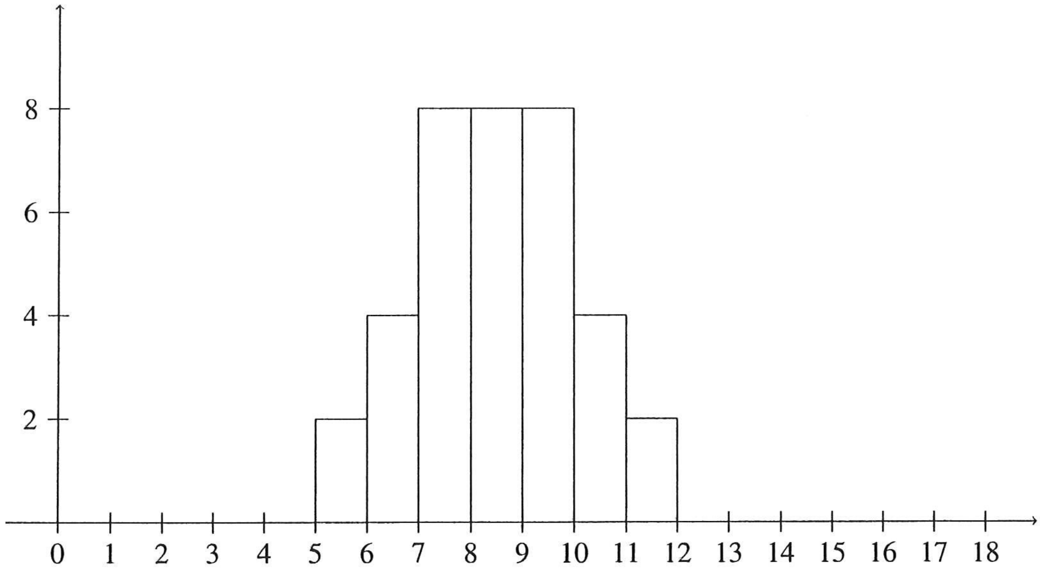


(iii) It often helps to draw up a table when performing histogram equalisation: below let $H(i)$ be the frequency values and $C(i)$ be the cumulative frequency values

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$H(i)$	0	0	0	0	0	2	4	8	8	8	4	2	0	0	0	0	0	0
$C(i)$	0	0	0	0	0	2	6	14	22	30	34	36	36	36	36	36	36	36

The transformed levels are given by

$$y_k = \sum_{i=1}^{k} L \frac{N_i}{NM}, \quad k = 1 \ldots 18$$

where $N \times M$ are the dimensions of the image, N_i is the number of pixels in grey level i (equivalent to $H(i)$ above) and L is the range in grey level space. Therefore, $L = 18$, $NM = 36$ and

$$y_k = \frac{L}{NM} \sum_{i=1}^{k} N_i = \frac{1}{2} \sum_{i=1}^{k} N_i = \tfrac{1}{2} C(k), \quad k = 1 \ldots 18$$

We can now add an extra line to our table to show the transformed values:

(cont.

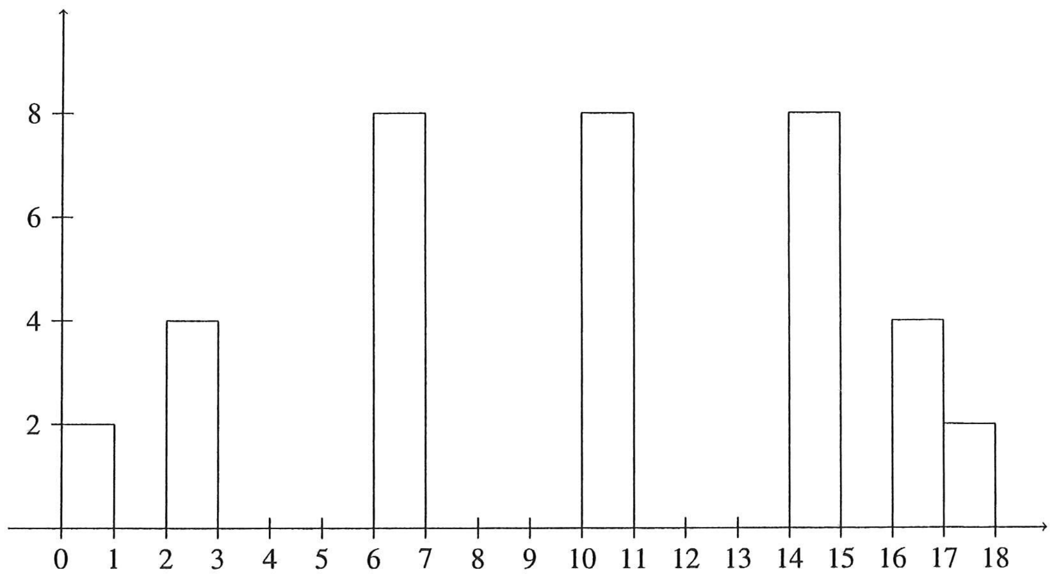
i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
$H(i)$	0	0	0	0	0	2	4	8	8	8	4	2	0	0	0	0	0	0
$C(i)$	0	0	0	0	0	2	6	14	22	30	34	36	36	36	36	36	36	36
$y(i)$	0	0	0	0	0	1	3	7	11	15	17	18	18	18	18	18	18	18

[20%]

From this table it is now easy to draw the new image and sketch the new histogram

3	7	17	11	3	7
11	1	15	15	18	7
17	7	17	11	1	3
15	7	15	15	17	11
11	18	15	15	11	7
7	3	11	15	7	11

Fig. 2



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We can see from the new histogram that the process has succeeded in spreading out the grey levels more evenly across the scale but that the distribution is far from being uniform. The discreteness of the problem means that the equalisation process tries to do the best job it can according to the rules prescribed. The spread of greylevels can then be improved by interpolation after the histogram equalisation process. [10%]

(b) Our observed image, $\mathbf{y}$, is modelled as a linear distortion, L, of the true image, $\mathbf{x}$, plus additive noise, $\mathbf{d}$, i.e. $\mathbf{y} = L\mathbf{x} + \mathbf{d}$.

(i) If we neglect noise we can write the above equation in discrete form as

$$y(n_1,n_2) = \sum_{m_1}\sum_{m_2} L(m_1,m_2)x(n_1-m_1,n_2-m_2)$$

Since the relationship between x and y is a 2-D convolution, a straightforward approach to the problem of reconstruction is to take the Fourier transform of each side of the above to give:

$$Y(\omega_1,\omega_2) = \mathcal{L}(\omega_1,\omega_2)X(\omega_1,\omega_2)$$

where:

$$\mathcal{L}(\omega_1,\omega_2) = \sum_{n_2=-\infty}^{\infty}\sum_{n_1=-\infty}^{\infty} L(n_1,n_2)e^{-j(\omega_1 n_1+\omega_2 n_2)}$$

$$\therefore X(\omega_1,\omega_2) = \frac{Y(\omega_1,\omega_2)}{\mathcal{L}(\omega_1,\omega_2)}$$

and

$$x(n_1,n_2) = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} X(\omega_1,\omega_2)e^{j(\omega_1 n_1+\omega_2 n_2)}d\omega_1 d\omega_2$$

Thus, if we neglect noise and know the psf, L, we can estimate our true image by a process known as *inverse filtering*, which, as we see above, involves dividing the fourier transform of the observed image by the fourier transform of L – the inverse filter is therefore $1/\mathcal{L}$.

If the transfer function $\mathcal{L}(\omega_1,\omega_2)$ has zeros then the inverse filter, $1/\mathcal{L}$, will have infinite gain. i.e. when $\mathcal{L}(\omega_1,\omega_2)$ is very small, $1/\mathcal{L}(\omega_1,\omega_2)$ is very large (or indeed infinite if there are zeros) and therefore,

(cont.

small noise in the regions of the frequency plane where $1/\mathscr{L}(\omega_1,\omega_2)$ is very large, can be hugely amplified. In practice a method of lessening this sensitivity to noise is to threshold the frequency response, leading to the so-called, pseudo-inverse or generalised inverse filter $\mathscr{L}_g(\omega_1,\omega_2)$. This is given by

$$\mathscr{L}_g(\omega_1,\omega_2) = \begin{cases} \frac{1}{\mathscr{L}(\omega_1,\omega_2)} & \frac{1}{|\mathscr{L}(\omega_1,\omega_2)|} < \gamma \\ 0 & \text{otherwise} \end{cases} \tag{2}$$

or

$$\mathscr{L}_g(\omega_1,\omega_2) = \begin{cases} \frac{1}{\mathscr{L}(\omega_1,\omega_2)} & \frac{1}{|\mathscr{L}(\omega_1,\omega_2)|} < \gamma \\ \gamma\frac{|\mathscr{L}(\omega_1,\omega_2)|}{\mathscr{L}(\omega_1,\omega_2)} & \text{otherwise} \end{cases} \tag{3}$$

Clearly for $\frac{1}{|\mathscr{L}(\omega_1,\omega_2)|} \geq \gamma$ in equation 3, the modulus of the filter is set as γ, whereas in equation 2 it is set as 0. [20%]

(ii) For simplicity we assume that $E[\mathbf{x}] = 0$ and $E[\mathbf{d}] = 0$, i.e. that both the signal and the noise are zero mean. To find an estimate of $\mathbf{x}$, we maximise $P(\mathbf{x}|\mathbf{y})$, i.e. the *probability* of the original image *given* the observed data. When dealing with *conditional probabilities* we use *Bayes' Theorem*:

$$P(\mathbf{x}|\mathbf{y}) = \frac{1}{P(\mathbf{y})}P(\mathbf{y}|\mathbf{x})P(\mathbf{x}) \tag{4}$$

at the simplest level we regard $P(\mathbf{y})$, the probability of the data, simply as a normalising factor, which therefore implies that we wish to maximise

$$P(\mathbf{x}|\mathbf{y}) \propto P(\mathbf{y}|\mathbf{x})P(\mathbf{x})$$

If we assume that the noise is gaussian distributed we can write the probability of the noise, which is proportional to the likelihood as

$$P(\mathbf{y}|\mathbf{x}) \propto \mathrm{e}^{-\frac{1}{2}\mathbf{d}^T N^{-1}\mathbf{d}} = \mathrm{e}^{-\frac{1}{2}(\mathbf{y}-L\mathbf{x})^T N^{-1}(\mathbf{y}-L\mathbf{x})}$$

where $N = E[\mathbf{d}\mathbf{d}^T]$ is the noise covariance matrix. The $\mathbf{d}^T N^{-1}\mathbf{d}$ term is the vector equivalent of the $\frac{1}{\sigma}$ term in the 1d gaussian – if N is diagonal then N^{-1} will be diagonal with elements $\frac{1}{\sigma_i}$.

We now have to decide on the assignment of the *prior* probability $P(\mathbf{x})$ – this probability incorporates any *prior* knowledge we may have about the distribution of the data.

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Assume first an ideal world in which $\mathbf{x}$ is a gaussian random variable, described by a *known* covariance matrix $C = E[\mathbf{x}\mathbf{x}^T]$ (including all cross-correlations etc.) so that

$$P(\mathbf{x}) \propto \mathrm{e}^{-\frac{1}{2}\mathbf{x}^T C^{-1}\mathbf{x}}$$

Thus we can now write the posterior probability as

$$P(\mathbf{x}|\mathbf{y}) \propto P(\mathbf{y}|\mathbf{x})P(\mathbf{x}) \propto \mathrm{e}^{-\frac{1}{2}[(\mathbf{y}-L\mathbf{x})^T N^{-1}(\mathbf{y}-L\mathbf{x})+\mathbf{x}^T C^{-1}\mathbf{x}]}$$

which one must maximise wrt $\mathbf{x}$ to obtain the reconstruction.

For *Maximum Entropy* deconvolution, again assuming Gaussian noise, the likelihood would be as for the Wiener filter above but the prior is an *entropic* prior which takes the form

$$P(\mathbf{x}) \propto \mathrm{e}^{\alpha S}$$

where one version of the *entropy* S (sometimes known as the *cross entropy*) of the image is given by

$$S(\mathbf{x},\mathbf{m}) = \sum_i \left[x_i - m_i - x_i \ln\left(\frac{x_i}{m_i}\right)\right]$$

where $\mathbf{m}$ is the *measure* on an image space (*the model*) to which the image $\mathbf{x}$ defaults in the absence of data. (Can see global maximum of S occurs at $\mathbf{x} = \mathbf{m}$.) [30%]

3(a) If the period is $\frac{2N}{k}$ samples, then the cosines must be of the form $A\cos\left(\frac{i.k.2\pi}{2N} + \phi_0\right)$ for $k = 0 \ldots N-1$.

For row 1 of the matrix

$$t_{1i} = A.\cos(0 + \phi_0) = B \qquad \text{for } i = 1 \ldots N$$

Hence for unit magnitude, $B = \frac{1}{\sqrt{N}}$. $\left(\sum_{i=1}^{N} t_{1i}^2 = 1\right)$

$$\therefore t_{1i} = \frac{1}{\sqrt{N}}$$

For row k of the matrix $(k = 2 \ldots N)$

$$t_{ki} = A\cos\left(\frac{i.(k-1).2\pi}{2N} + \phi_0\right)$$

To get symmetry/anti-symmetry about the mid points, ϕ_0 must correspond to $-\frac{1}{2}$ sample offset in i so that $(i-\frac{1}{2})$ goes from $\frac{1}{2}$ to $N-\frac{1}{2}$ as i goes from 1 to N,

Hence $\phi_0 = -\frac{1}{2}.\frac{(k-1).2\pi}{2N}$

$$\therefore t_{ki} = A\cos\left(\frac{(i-\frac{1}{2})(k-1).2\pi}{2N}\right) = A\cos\left(\frac{(2i-1)(k-1)\pi}{2N}\right)$$

for $i = 1 \ldots N$.

The mean square value of a uniformly sampled cosine wave is $\frac{1}{2}$ (because $\cos 2\theta$ averages to zero).

Hence $A = \sqrt{\frac{2}{N}}$ for unit magnitude of each row.

$$\therefore t_{ki} = \sqrt{\frac{2}{N}}\cos\left(\frac{(2i-1)(k-1)\pi}{2N}\right)$$

[25%]

3(b) 4-point DFT of $\underline{x} = \begin{bmatrix} p \\ p \\ q \\ q \end{bmatrix}$

$\underline{y} = T\underline{x}$

$\therefore y_k = \sum_{i=1}^{4} t_{ki} \, x_i$

$\therefore y_1 = \sqrt{\frac{1}{4}}\,(2p + 2q) = \underline{p + q}$

$y_2 = \sqrt{\frac{2}{4}}\left[p\left(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8}\right) + q\left(\cos\frac{5\pi}{8} + \cos\frac{7\pi}{8}\right)\right]$

$= \frac{1}{\sqrt{2}}(p-q)\left(\cos\frac{\pi}{8} + \cos\frac{3\pi}{8}\right) = \underline{0.9239(p-q)}$

$y_3 = \sqrt{\frac{2}{4}}\left[p\left(\cos\frac{\pi}{4} + \cos\frac{3\pi}{4}\right) + q\left(\cos\frac{5\pi}{4} + \cos\frac{7\pi}{4}\right)\right]$

$= \underline{\underline{0}}$

$y_4 = \sqrt{\frac{2}{4}}\left[p\left(\cos\frac{3\pi}{8} + \cos\frac{9\pi}{8}\right) + q\left(\cos\frac{15\pi}{8} + \cos\frac{21\pi}{8}\right)\right]$

$= \frac{1}{\sqrt{2}}(p-q)\left(\cos\frac{3\pi}{8} - \cos\frac{\pi}{8}\right) = \underline{-0.3827(p-q)}$

[20%]

3(c) $\underline{y} = T\underline{x}$ transforms 1 column-vector $\underline{x}$

$\therefore$ TX transforms all N columns of X

$\therefore$ $(TX^T)^T = XT^T$ transforms all N rows of X

$\therefore$ $Y = \underline{\underline{TXT^T}}$ " all rows & cols of X

Since T is orthonormal, $T^{-1} = T^T$

$\therefore T^T Y T = T^T T X T^T T = I X I = X$

Hence $\underline{\underline{T^T Y T}}$ recovers X from Y

[20%]

3(d) We need $Y = T X T^T$

Consider $W = TX$ first:

For first 2 columns, $p = q = r = 50$.

Using results from part (b):

$$\therefore \underline{W}_1 = \underline{W}_2 = \begin{bmatrix} p+q \\ 0.9239(p-q) \\ 0 \\ -0.3827(p-q) \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

For last 2 columns, $p = r = 50$ & $q = s = 150$

$$\underline{W}_3 = \underline{W}_4 = \begin{bmatrix} 200 \\ -92.39 \\ 0 \\ +38.27 \end{bmatrix}$$

Now $Y = TXT^T = WT^T = (TW^T)^T$

$$W^T = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 100 & 0 & 0 & 0 \\ 200 & -92.39 & 0 & 38.27 \\ 200 & -92.39 & 0 & 38.27 \end{bmatrix}$$

Hence

$$TW^T = \begin{bmatrix} 300 & -92.39 & 0 & 38.27 \\ -92.39 & 85.36 & 0 & -35.36 \\ 0 & 0 & 0 & 0 \\ 38.27 & -35.36 & 0 & 14.64 \end{bmatrix}$$

$$\therefore Y = (TW^T)^T = \underline{\underline{TW^T}}$$ since it is symmetric.

3(d) To encode Y efficiently, it would first be quantised - eg. with stepsize = 15 it would become

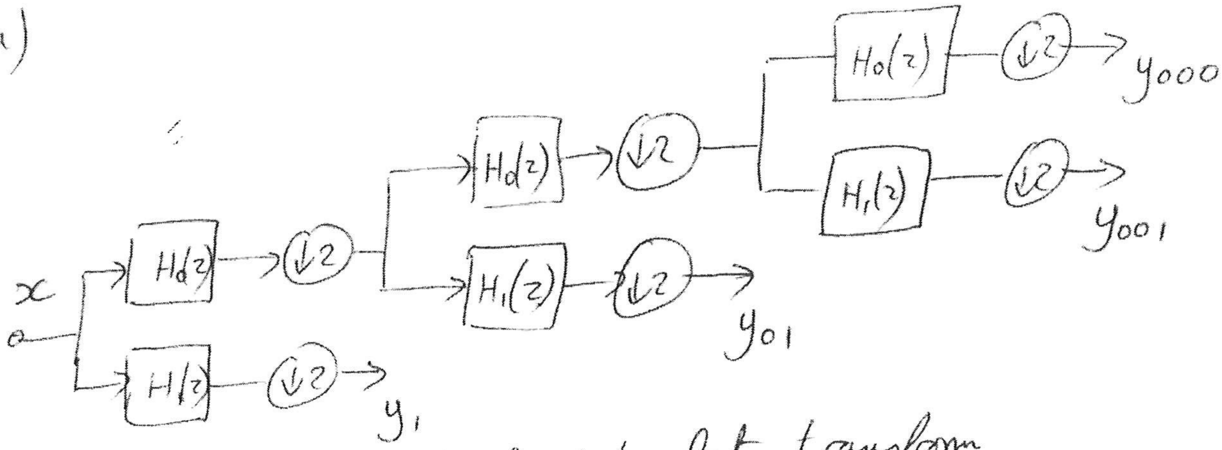
$$\begin{bmatrix} 20 & -6 & 0 & 2 \\ -6 & 6 & 0 & -2 \\ 0 & 0 & 0 & 0 \\ 2 & -2 & 0 & 1 \end{bmatrix}$$

Then it would be scanned in zig-zag order to obtain [20 -6 -6 0 6 0 2 0 0 2 -2 0 -2 0 0 1]

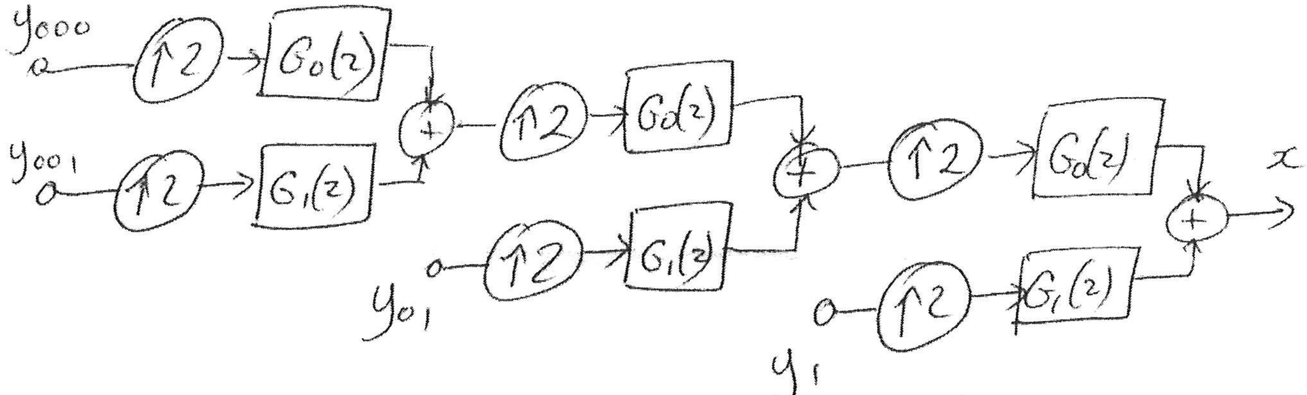
and combined (run-length, ~~size~~ value) Huffman coding would be used to create an efficient binary code for all the non-zero symbols. The zeros are coded in the run-length information. The 'value' is often split into (size, additional bits) where the 'size' is Huffman coded, and the additional bits are ~~linearly~~ uniformly coded, with a word-length equal to 'size'.

[35%]

4 (a)

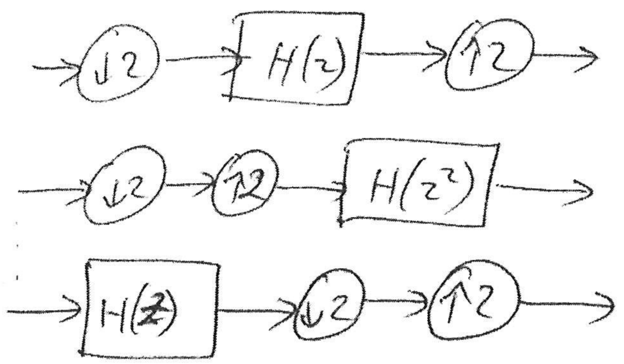


3-level Wavelet transform.



3-level inverse Wavelet transform. [15%]

4 (b)



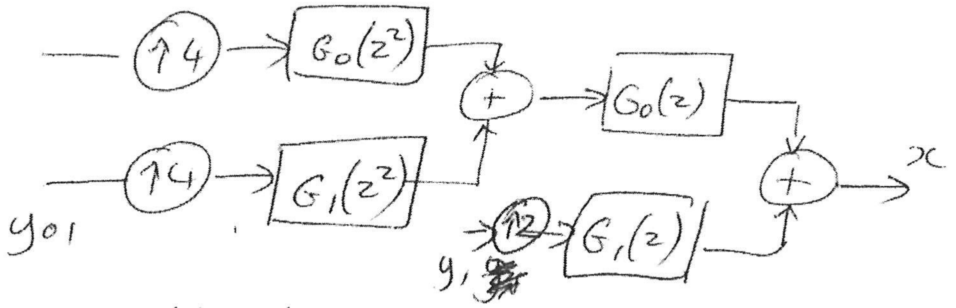
is equivalent to

or

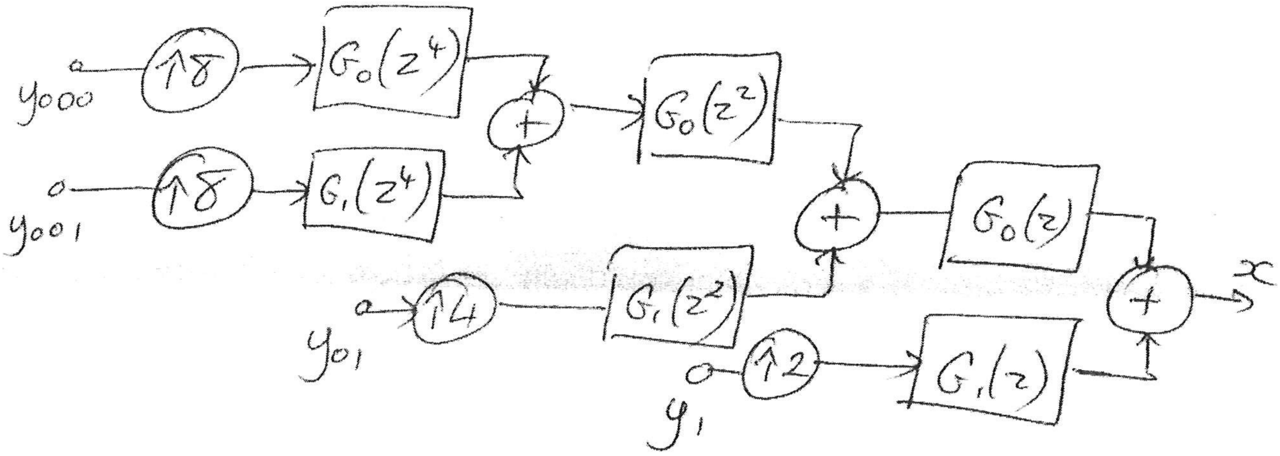
Hence if the filter $H(z)$ is moved from being ahead of the ~~down~~upsampler (upper diag.) to being after the upsampler, then $H(z)$ is replaced by $H(z^2)$.

[10%]

4(c) Moving the 2:1 upsamplers of the final stage in part (a) ahead of the second stage produces



Similarly moving the 4:1 upsampler ahead of the filters in the first stage produces



Hence the transfer function from y_{000} to x is

$$G_0(z^4) . G_0(z^2) . G_0(z)$$

and from y_{001} to x is

$$G_1(z^4) . G_0(z^2) . G_0(z)$$

[20%]

(d) Using the given filters

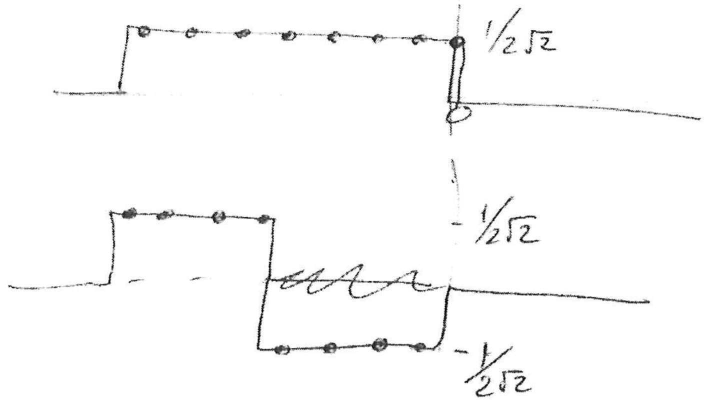
$$G_0(z^2) . G_0(z) = \frac{1}{2}(z^2+1)(z+1) = \frac{1}{2}(z^3+z^2+z+1)$$

So $G_0(z^4) . G_0(z^2) . G_0(z) = \frac{1}{2\sqrt{2}}(z^7+z^6+z^5+z^4+z^3+z^2+z+1)$

& $G_1(z^4) . G_0(z^2) . G_0(z) = \frac{1}{2\sqrt{2}}(z^7+z^6+z^5+z^4-z^3-z^2-z-1)$

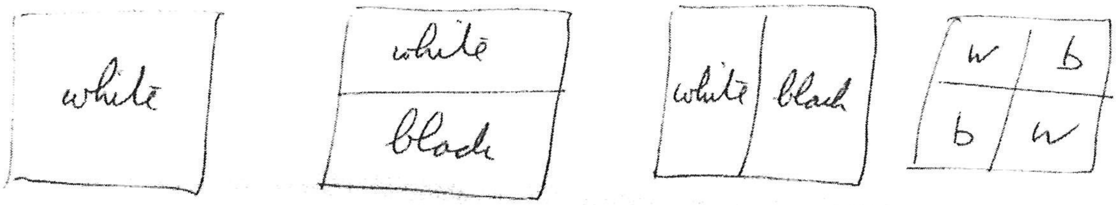
The upsamplers at the inputs do not affect the impulse responses as an impulse is not affected by the up sampling.

4 (d)



[20%]

(e) ~~The~~ When extended to 2-D, these two functions produce the following 4 types of image patch, each of size 8 × 8 pixels:



If the ~~higher the~~ finer level wavelet coefs are small enough to be quantised to zero, then the image will be comprised of linear combinations of patches as shown, & ~~will~~ the sharp edges will be visible as blocking artefacts.

[15%]

(f) To eliminate blocking, the basis functions must decay smoothly to zero at their end points. The simplest filter to achieve this is

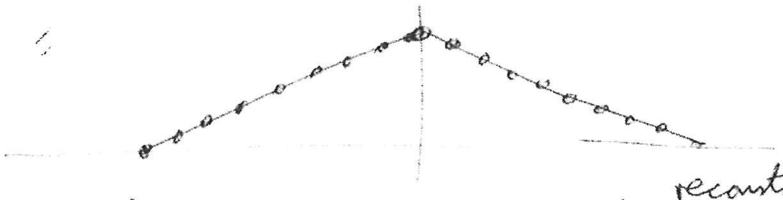
$$G_0(z) = K(z + 2 + z^{-1})$$

This gives

$$G_0(z^2).G_0(z) = K^2(z^3 + 2z^2 + 3z + 4 + 3z^{-1} + 2z^{-2} + z^{-3})$$

which linearly interpolates the outputs of $G_0(z^4)$ & $G_1(z^4)$.

4(f) Hence the lowpass basis function becomes



This will give much smoother ~~responses~~ reconstructed image regions, & is ~~known to~~ part of the LeGall filter system.

In 2-D these triangular functions become pyramid-like structures which decay smoothly to zero at their edges. [20%]