DR T P HYNES

Part II B
Part II A

2010 4M12 Crib

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

1. (a) $\nabla^4 w = \nabla.\nabla(\nabla.\nabla w) = \nabla.\nabla \dfrac{\partial^2 w}{\partial x_i \partial x_i}$

$$= \frac{\partial^2}{\partial x_j \partial x_j}\frac{\partial^2 w}{\partial x_i \partial x_i} = \frac{\partial^4 w}{\partial x_j \partial x_j \partial x_i \partial x_i}$$

This is symmetric $i \leftrightarrow j$

(b) $\epsilon_{ipq}\epsilon_{l\alpha q} = \delta_{il}\delta_{p\alpha} - \delta_{i\alpha}\delta_{lp}$ (using datacard formula)

$\therefore \epsilon_{ipq}\epsilon_{lpq} = \delta_{il}\delta_{pp} - \delta_{ip}\delta_{lp}$

$= 3\delta_{il} - \delta_{il}$ $\left(\delta_{pp} = \delta_{11} + \delta_{22} + \delta_{33} = 1+1+1 = 3\right)$

$= 2\delta_{il}$

(c) $(\underline{w} \times \underline{v})_i = \epsilon_{ijk} w_j v_k$

$= W_{ik} v_k$ where $W_{ik} = \epsilon_{ijk} w_j$

$= (W\underline{v})_i$

$\underline{v}$ arbitrary $\Rightarrow$ W unique

Also $W_{ki} = \epsilon_{kji} w_j = \epsilon_{ikj} w_j$ (ϵ_{ijk} unchanged by cyclic permutation of i,j,k)

$= -\epsilon_{ijk} w_j$ (ϵ changes sign if i,j,k move between cyclic & "anticyclic")

$= -W_{ik}$

$\therefore$ W is skew symmetric

(d) $$J(u+\varepsilon\bar{u}) = \int_V (\nabla^2 u + \varepsilon\nabla^2\bar{u})(\nabla^2 u + \varepsilon\nabla^2\bar{u})\,dV$$

Suppose u corresponds to a stationary point; it will be a <u>minimum</u> if $\left.\dfrac{\partial^2 J}{\partial\varepsilon^2}\right|_{\varepsilon=0} \geqslant 0$ for any $\bar{u}$.

$$\frac{\partial^2 J}{\partial\varepsilon^2} = 2\int_V (\nabla^2\bar{u})^2\,dV$$ which ≥ 0 since integrand is the square of something.

<u>Examiner's Note</u>

All candidates did well on parts (a) – (c). For reasons that are beyond me, the vast majority of candidates did not read part (d) properly and produced (correct) derivations of the conditions to make J stationary, ignoring whether it is a max, min or neither.

2(a) u minimises the functional if $\left.\frac{\partial J(u+\varepsilon\bar{u})}{\partial \varepsilon}\right|_{\varepsilon=0} = 0$ for all $\bar{u}$.

Now $J(u+\varepsilon\bar{u}) = \frac{1}{2}\int[(\nabla u+\varepsilon\nabla\bar{u}).(\nabla u+\varepsilon\nabla\bar{u}) - f(u+\varepsilon\bar{u})]dV$

$$\left.\frac{\partial J}{\partial \varepsilon}\right|_{\varepsilon=0} = \int_V (\nabla\bar{u}.\nabla u - f\bar{u})\,dV$$

$$= \int_V \nabla.(\bar{u}\nabla u) - \bar{u}\nabla^2 u - f\bar{u}\,dV$$

$$= \int_{\partial V} \bar{u}\,\underline{n}.\nabla u\,dS - \int_V \bar{u}(\nabla^2 u + f)\,dV$$

For this to be zero for arbitrary $\bar{u}$, we need

(i) $-\nabla^2 u = f$

(ii) $\bar{u}$ (& hence u) $= 0$ on ∂V

or $\underline{n}.\nabla u = 0$ on the boundary

In the latter case, need u (& $\bar{u}$) $= 0$ at some point on ∂V.

(b) (i)

$$J = \frac{1}{2}\int (u-w)^2 + f^2\,dV + \int \lambda(\nabla^2 u + f)\,dV$$

(ii) $J = J(u, f, \lambda)$

Stationary point when A) $\left.\frac{\partial}{\partial \varepsilon} J(u+\varepsilon\bar{u}, f, \lambda)\right|_{\varepsilon=0} = 0$ for all $\bar{u}$

B) $\left.\frac{\partial}{\partial \varepsilon} J(u, f+\varepsilon\bar{f}, \lambda)\right|_{\varepsilon=0} = 0$ for all $\bar{f}$

and C) $\left.\frac{\partial J}{\partial \varepsilon}(u, f, \lambda+\varepsilon\bar{\lambda})\right|_{\varepsilon=0} = 0$ for all $\bar{\lambda}$

Now $\quad J(u+\varepsilon\bar{u}, f, \lambda) = \frac{1}{2}\int (u+\varepsilon\bar{u}-w)^2 + f^2 \, dV$

$\qquad\qquad + \int \lambda(\nabla^2(u+\varepsilon\bar{u}) + f)\, dV$

(A) $\Rightarrow \quad \int (\bar{u}(u-w) + \lambda \nabla^2 \bar{u})\, dV = 0 \qquad (*)$

(B) $\Rightarrow \quad \int (f\bar{f} + \lambda\bar{f})\, dV = 0 \quad \Rightarrow f = -\lambda$ since $\bar{f}$ arbitrary

(C) $\Rightarrow \quad \int (\nabla^2 u + f)\, dV = 0 \quad \Rightarrow -\nabla^2 u = f$

Returning to $(*)$

$0 = \int_V [\lambda \nabla^2 \bar{u} + (u-w)\bar{u}]\, dV$

Now $\int (\bar{u}\nabla^2\lambda - \lambda\nabla^2\bar{u})\, dV = \int_{\partial V} \Big(\underbrace{\bar{u}\frac{\partial\lambda}{\partial n}}_{0 \text{ since } \bar{u}=0 \text{ on } \partial V} - \lambda\frac{\partial \bar{u}}{\partial n}\Big)\, dS$

$\therefore \quad \int_V [\bar{u}\nabla^2\lambda + (u-w)\bar{u}]\, dV + \int_{\partial V} \lambda \frac{\partial \bar{u}}{\partial n}\, dS = 0$

Since $\bar{u}$ arbitrary we need

$-\nabla^2\lambda = u - w$ in V

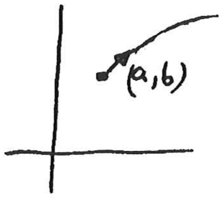
and $\quad \lambda = 0$ on ∂V

Examiner's Note

Part (a) was very well done.

Most candidates had trouble with part (b). Common errors were: treating λ as indept of position and taking it outside the integral in J; trying to use part (a) too much.

3 (a) $\quad a\dfrac{\partial u}{\partial x} + b\dfrac{\partial u}{\partial y} = c \quad$ is equivalent to $(a,b).\nabla u = c$



$(a,b).\nabla u$ is the derivative in the direction (a,b). This equation is thus one for the rate of change in a so-called characteristic direction $\parallel (a,b)$. If ξ is introduced as a variable that varies along the line derived by following the (a,b) direction, then the equation becomes an o.d.e. for u

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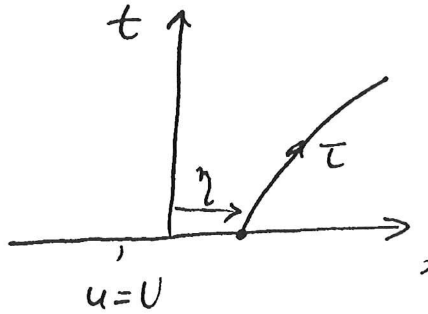
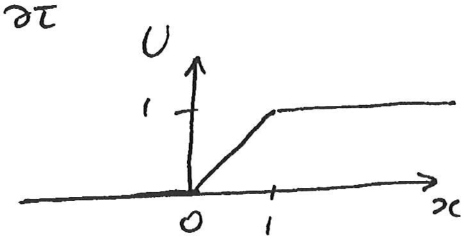
(b) $\quad \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} = 2$



Introduce τ as ch'ic variable & η as label for ch'ic

$\Rightarrow \quad \dfrac{\partial t}{\partial \tau} = 1 \ \text{(i)} \quad \dfrac{\partial x}{\partial \tau} = u \ \text{(ii)}$

$\Rightarrow \quad \dfrac{\partial t}{\partial \tau}\dfrac{\partial u}{\partial t} + \dfrac{\partial x}{\partial \tau}\dfrac{\partial u}{\partial x} = 2 \quad$ i.e. $\quad \dfrac{\partial u}{\partial \tau} = 2 \ \text{(iii)}$



(iii) $\Rightarrow \quad u = 2\tau + U(\eta)$

where η = initial value of x on ch'ic. (i.e. at $t=0$)

$\dfrac{\partial t}{\partial \tau} = 1$ and $\dfrac{\partial x}{\partial \tau} = u = 2\tau + U \quad \Rightarrow \quad t = \tau$ and $x = \tau^2 + U\tau + \eta$

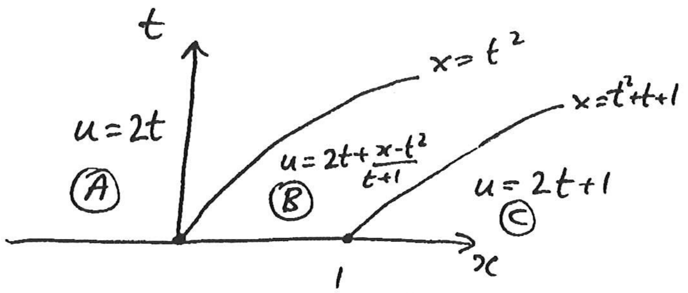
(using $t|_{\tau=0} = 0$ & $x|_{\tau=0} = \eta$)

So $\quad u(x,t) = U(\eta) + 2t \quad$ along a line $\quad x = t^2 + U(\eta)t + \eta$

where $U(\eta) = \begin{cases} 0 & \eta < 0 \quad \text{on } x = t^2 + \eta \\ \eta & 0 < \eta < 1 \quad \text{on } x = t^2 + (t+1)\eta \\ 1 & \eta > 1 \quad \text{on } x = t^2 + t + \eta \end{cases}$

Eliminating η

$$u(x,t) = \begin{cases} 2t & \text{for } x - t^2 < 0 \\ 2t + \dfrac{x-t^2}{t+1} & 0 < \dfrac{x-t^2}{t+1} < 1 \\ 2t+1 & x - t^2 - t > 1 \end{cases}$$



(c) When the shape of the characteristic lines depend on the solution u, there is always the possibility that "faster" moving ones will overtake "slower" moving ones and then discontinuities form

In this case ch'ics are parallel in regions A & C and they are spreading out for $0 < \eta < 1$ in B

$\Rightarrow$ no discontinuities form.

<u>Aliter</u>

ch'ic 1 in B is $x = t^2 + \eta_1 t + \eta_1$

" 2 in C is $x = t^2 + \eta_2 t + \eta_2$

If these cross then $\eta_1 t + \eta_1 = \eta_2 t + \eta_2 \Rightarrow \eta_1 = \eta_2$

i.e. must be same ch'ic $\Rightarrow$ no discontinuities form

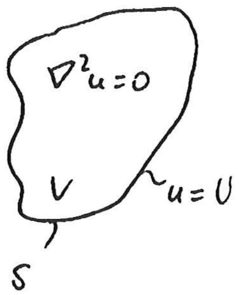
<u>Examiner's Note</u>

Most candidates knew what they were doing but a distressing number could not solve the 3 o.d.e. for u, x & t. The commonest error was forgetting that u depends on τ and saying

$$\frac{\partial x}{\partial \tau} = u \quad \Rightarrow \quad x = u\tau + \text{const}$$

4(a) A problem is well posed if
(i) the solution exists
(ii) is unique
(iii) depends continuously on the data
(i.e. is 'robust' or 'stable' to changes to b.c. data so that if this changes slightly, then the solution will also only change slightly).

(b) (i) The maximum principle for $\nabla^2 u = 0$ states that the maximum and minimum of u occur on the boundary.



This means that any changes to u in the interior are automatically smaller than changes in U.

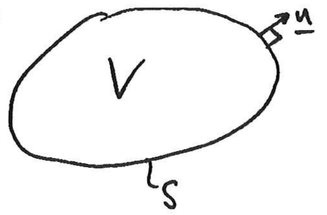
(ii) if u_1 & u_2 are solutions then

$$\nabla^2(u_1 - u_2) = 0 \text{ in } V \quad \text{and } u_1 - u_2 = U - U = 0 \text{ on } S$$

The max & min values of $u_1 - u_2$ are thus zero i.e. $u_1 \equiv u_2$

This means that the solution, if it exists, is unique.

(c)



$$\int \nabla^2 \varphi \, dV = \int f \, dV$$
$$= \int \frac{\partial \varphi}{\partial n} dS \quad \text{using Gauss' Thm}$$

For solutions to be possible, therefore, we must have

$$\int \psi \, dS = \int f \, dV.$$

In addition, we can add a constant to φ and not change the equation or boundary condition. $\therefore$ Need φ specified at some point on S to make solution unique.

(d) (i) Put $W d\underline{w} = d\underline{r}$ for some W to be found later

$$\Rightarrow \quad W \frac{\partial \underline{w}}{\partial t} + A W \frac{\partial \underline{w}}{\partial x} = 0$$

or
$$\frac{\partial \underline{w}}{\partial t} + W^{-1} A W \frac{\partial \underline{w}}{\partial x} = 0 \qquad (1)$$

If $W^{-1}AW$ can be chosen to be diagonal, then (1) will take the form
$$\frac{\partial \underline{w}}{\partial t} + \begin{bmatrix} \lambda^{(1)} & 0 & 0 \\ 0 & \lambda^{(2)} & 0 \\ 0 & 0 & \lambda^{(3)} \end{bmatrix} \frac{\partial \underline{w}}{\partial x} = 0 \qquad \Rightarrow \quad AW = \Lambda W$$

i.e.
$$\frac{\partial w_i}{\partial t} + \lambda^{(i)} \frac{\partial w_i}{\partial x} = 0 \qquad \text{(N.B. No summation convention!)}$$

which can be integrated along characteristics.

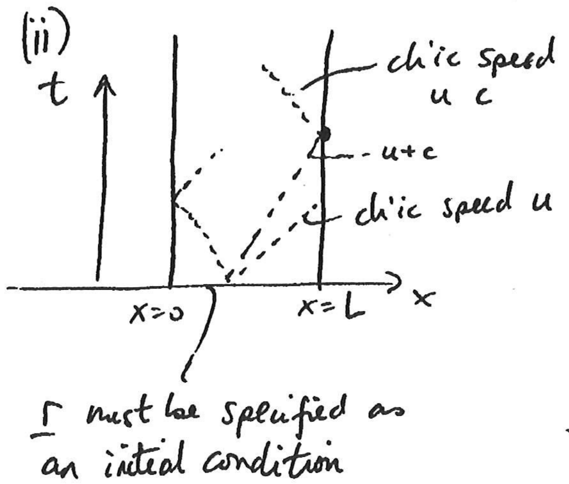
The equations will thus be hyperbolic if all the e-values of A are real. These are given by

$$0 = \det(A - \lambda I) = \begin{vmatrix} u-\lambda & c & 0 \\ c & u-\lambda & 0 \\ 0 & 0 & u-\lambda \end{vmatrix} = [(u-\lambda)^2 - c^2][u-\lambda]$$

i.e. $\lambda = u$ or $\lambda = u + c$ or $\lambda = u - c$

These are the <u>characteristic speeds</u>

(ii)



B.C.'s must be compatible with ch'ic directions.

$u<c$ { i.e. one component must be specified at $x=L$ and 2 specified at $x=0$

for $u>c$

Need 3 b.c.'s at $x=0$

<u>Examiner's Note</u>: Pleasingly, nearly all candidates answered this question well.

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30/5/10