Solutions for 4M13 Section A

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1. a) The correct mapping is $w = u + iv = G(z) = z^2$. The derivative is $G'(z) = 2z = 2(x + iy)$.

   b) In the mapped space, the streamlines of the flow in the upper half plane are straight lines parallel to the x axis, with constant velocity V. So the velocity field is $(v_u, v_v) = \nabla\phi(u, v) = (V, 0)$, giving a potential function $f(w) = Vu + iVv = Vw$. The velocity field in the original space is then given by $v_x - iv_y = (v_u - iv_v)G'(z)$, giving $v_x = 2Vx$ and $v_y = -2Vy$. The stream function is given by the relation $\nabla\psi = (-v_y, v_x)$, so $\psi = 2Vxy$. The real part of the potential function is $\phi = V(x^2 - y^2)$.

   c) The above velocity field is unbounded as $x \to \infty$, so clearly cannot be valid. The flow is valid as given close to the corner. This is also obvious from the hyperbolic streamlines, since they would imply an ever growing amount of fluid flowing through a given cross section as $x \to \infty$. In practice, the flow field is a good approximation for the flow of a finite width jet hitting a wall (or corner).

Assessor's comment:

The unpopularity perhaps is because a question on this topic has not appeared in recent Tripos papers, although it has always been on the syllabus. Of those who attempted it, 4 did quite well, and 2 hardly got into it. In retrospect, the last part of the question was too easy, fewer clues could have been given.

2. a) The integral is the real part of

$$J = \int_{-\infty}^{\infty} \frac{e^{i\lambda z}}{z^2+1}$$

Complete the contour by a large semicircle. There are poles at $z = i$ and $z = -i$. Jordan's Lemma implies the integral vanishes on the semicircle, so the integral on the real line is the sum of the residue in the UHP ($\times 2\pi i$),

$$I = 2\pi i \lim_{z\to i} \frac{e^{i\lambda z}(z-i)}{(z+i)(z-i)} = \pi e^{-\lambda}$$

b) The integral is even, so we can extend it to the entire real axis. It is then half of the imaginary part of the complex version,

$$I = \frac{1}{2}\mathcal{I}\left\{\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz\right\}$$

Complete the contour with a large semicircle in the UHP. Jordan's Lemma implies that the integral vanishes on the semicircle. There is a simple pole at $z = 0$, which is on the contour, so its contribution is $\pi i \times$ residue. The residue is

$$\lim_{z\to 0} z\frac{e^{iz}}{z} = 1.$$

So

$$I = \frac{1}{2}\mathcal{I}\{\pi i\} = \pi/2$$

Assessor's comment:

All candidates attempted this question. It was pleasant to see a number of perfect solutions. The first integral seemed to be slightly easier for the candidates than the second one, in the latter the most common mistake was to try and claim that the semicircular part of the complex integral vanishes for the sin(z)/z integrand, instead of separating it into exponentials, and then making the claim using Jordan's Lemma.

Question 3

(a)

Inequalities of the form $\underline{a} \cdot \underline{x} \leq b$ can be made into equality constraints by introducing a new slack variable x_s such that $\underline{a} \cdot \underline{x} + x_s = b$.

Similarly, inequalities of the form $\underline{a} \cdot \underline{x} \geq b$ can be made into equality constraints by introducing a new slack variable x_s such that $\underline{a} \cdot \underline{x} - x_s = b$. [10%]

(b)

Let x_1, x_2, x_3 be the number of opencast, mechanized and labour-intensive mines respectively.

The objective function (to be minimized) is

$$f = -10x_1 - 5x_2 + x_3$$

subject to (use of workers)

$$2x_1 + 5x_2 + 10x_3 = 70$$

and (coal production from labour-intensive mines)

$$x_3 \geq 5$$

and (use of cutting machines)

$$4x_1 + 2x_2 + x_3 \leq 50$$

The second constraint can be converted to standard form by introducing a slack variable x_4:

$$x_3 - x_4 = 5$$

The third constraint can be converted to standard form by introducing a slack variable x_5:

$$4x_1 + 2x_2 + x_3 + x_5 = 50$$

Thus, we have

$$\text{Minimise } f = -10x_1 - 5x_2 + x_3$$

$$\text{subject to } 2x_1 + 5x_2 + 10x_3 = 70$$

$$\text{and } x_3 - x_4 = 5$$

$$\text{and } 4x_1 + 2x_2 + x_3 + x_5 = 50$$

[20%]

(c)

The number of marks available implies that it is not necessary to use phase 1 of the simplex method to find a feasible initial solution. A few moments thought shows that

$$(x_1, x_2, x_3, x_4, x_5) = (0,0,7,2,43)$$

is a suitable feasible initial solution. (To be suitable as a starting point for the Simplex method, the initial solution must be a vertex of the feasible set.) [10%]

(d)

The initial tableau is

$$\begin{bmatrix} 2 & 5 & 10 & 0 & 0 & 70 \\ 0 & 0 & 1 & -1 & 0 & 5 \\ 4 & 2 & 1 & 0 & 1 & 50 \\ -10 & -5 & 1 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} \text{Row 1} \\ \text{Row 2} \\ \text{Row 3} \\ \text{Row 4} \end{matrix}$$

For the initial feasible solution identified, x_3, x_4 and x_5 are the basic variables, hence in canonical form the tableau is:

$$\begin{bmatrix} 0.2 & 0.5 & 1 & 0 & 0 & 7 \\ 0.2 & 0.5 & 0 & 1 & 0 & 2 \\ 3.8 & 1.5 & 0 & 0 & 1 & 43 \\ -10.2 & -5.5 & 0 & 0 & 0 & -7 \end{bmatrix} \begin{matrix} \text{Row 5} = 0.1 \times \text{Row 1} \\ \text{Row 6} = \text{Row 5} - \text{Row 2} \\ \text{Row 7} = \text{Row 3} - \text{Row 5} \\ \text{Row 8} = \text{Row 4} - \text{Row 5} \end{matrix}$$

Hence the entering variable is x_1. The leaving variable is x_4 : 2÷0.2 being less than 43÷3.8 or 7÷0.2. Hence the new tableau in canonical form is:

$$\begin{bmatrix} 0 & 0 & 1 & -1 & 0 & 5 \\ 1 & 2.5 & 0 & 5 & 0 & 10 \\ 0 & -8 & 0 & -19 & 1 & 5 \\ 0 & 20 & 0 & 51 & 0 & 95 \end{bmatrix} \begin{matrix} \text{Row 9} = \text{Row 5} - \text{Row 6} \\ \text{Row 10} = 5 \times \text{Row 6} \\ \text{Row 11} = \text{Row 7} - 19 \times \text{Row 6} \\ \text{Row 12} = \text{Row 8} + 10.2 \times \text{Row 10} \end{matrix}$$

As all the reduced costs are positive this means this is the optimum, with the number of opencast mines $x_1 = 10$ and the number of labour-intensive mines $x_3 = 5$.

The objective function $f = -95$ (minus the bottom right tableau value) and hence the profit is £95M per year. [50%]

(e)
At the optimum $x_5 = 5$, where x_5 is the slack variable associated with the use of cutting machines. This means that only 45 cutting machines are actually used for the optimal solution and the optimum will not be constrained by their availability until the number available falls below 45. [10%]

Assessor's comment:

A very popular question though disappointingly done. Not many candidates could do all of (a) set up a linear program in standard form, (b) remember the Simplex method, and (c) execute it without error. Common mistakes included using slack variables incorrectly and trying to be clever by anticipating (wrongly) features of the optimal solution. Quite a few candidates interpreted the request to identify a "suitable feasible initial solution" as a request to find any feasible solution and started the Simplex method from a solution that wasn't a vertex, with inevitable consequences.

Question 4

(a)

The total production of biofuels $= \sum_{i=1}^{n} a_i \sqrt{x_i h_i}$

The capital investment must be spent $\Rightarrow \sum_{i=1}^{n} x_i = k$

Thus, the problem is to

$$\text{Minimize} \quad f = -\sum_{i=1}^{n} a_i \sqrt{x_i h_i}$$

$$\text{Subject to} \quad \sum_{i=1}^{n} x_i - k = 0$$

[10%]

(b)

The Lagrangian is

$$L = -\sum_{i=1}^{n} a_i \sqrt{x_i h_i} + \lambda\left(\sum_{i=1}^{n} x_i - k\right)$$

Thus, the first-order optimality conditions are

$$\frac{\partial L}{\partial x_i} = -\frac{a_i}{2}\sqrt{\frac{h_i}{x_i}} + \lambda = 0 \quad \forall i \qquad (1)$$

$$\sum_{i=1}^{n} x_i - k = 0 \qquad (2)$$

From (1)

$$x_i = \frac{a_i^2 h_i}{4\lambda^2} \qquad (3)$$

Substituting in (2)

$$\sum_{i=1}^{n} \frac{a_i^2 h_i}{4\lambda^2} - k = 0$$

$$\lambda^2 = \sum_{i=1}^{n} \frac{a_i^2 h_i}{4k}$$

So, substituting in (3)

$$x_i = \frac{k a_i^2 h_i}{\sum_{i=1}^{n} a_i^2 h_i}$$

The second-order optimality conditions also need to be checked.

Differentiating (1)

$$\frac{\partial^2 L}{\partial x_i^2} = \frac{a_i}{4}\sqrt{\frac{h_i}{x_i^3}} \quad \forall i$$

and

$$\frac{\partial^2 L}{\partial x_i x_j} = 0 \quad \forall i \neq j$$

By inspection $\frac{\partial^2 L}{\partial x_i^2}$ will be positive for all physically feasible values of x_i. As the off-diagonal elements of the Hessian of L are zero, this means that the Hessian is *positive definite* for all feasible directions (indeed for all directions) and thus the solution found does represent a minimum. [40%]

(c)

The profit made from the investment in bonds $= \alpha b$

The profit made from biofuel production $= p\sum_{i=1}^{n} a_i\sqrt{x_i h_i}$

Thus, the problem is now to Minimize $f = -\alpha b - p\sum_{i=1}^{n} a_i\sqrt{x_i h_i}$

Subject to $b + \sum_{i=1}^{n} x_i - k = 0$

Now the Lagrangian is $L = -\alpha b - p\sum_{i=1}^{n} a_i\sqrt{x_i h_i} + \lambda\left(b + \sum_{i=1}^{n} x_i - k\right)$

Thus, the first-order optimality conditions are

$$\frac{\partial L}{\partial x_i} = -\frac{pa_i}{2}\sqrt{\frac{h_i}{x_i}} + \lambda = 0 \quad \forall i \tag{4}$$

$$\frac{\partial L}{\partial b} = -\alpha + \lambda = 0 \tag{5}$$

$$b + \sum_{i=1}^{n} x_i - k = 0 \tag{6}$$

From (4) $$x_i = \frac{p^2 a_i^2 h_i}{4\lambda^2}$$

From (5) $\alpha = \lambda$ $$\therefore\ x_i = \frac{p^2 a_i^2 h_i}{4\alpha^2}$$

From (6) $$b = k - \sum_{i=1}^{n} x_i = k - \sum_{i=1}^{n} \frac{p^2 a_i^2 h_i}{4\alpha^2} = k - \frac{p^2}{4\alpha^2}\sum_{i=1}^{n} a_i^2 h_i$$

Again, the second-order optimality conditions should be checked.

Differentiating (4) $$\frac{\partial^2 L}{\partial x_i^2} = \frac{pa_i}{4}\sqrt{\frac{h_i}{x_i^3}} \quad \forall i$$

and $$\frac{\partial^2 L}{\partial x_i x_j} = 0 \quad \forall i \neq j$$

Differentiating (5) $$\frac{\partial^2 L}{\partial b^2} = 0 = \frac{\partial^2 L}{\partial b \partial x_i}$$

And the same arguments apply as in part (b), noting that in this case one of the on-diagonal terms of the Hessian is zero. [50%]

Note that it is also possible to use the constraint to eliminate b from the expression for the objective function, i.e. substitute $b = k - \sum_{i=1}^{n} x_i$ giving $f = -\alpha k + \alpha\sum_{i=1}^{n} x_i - p\sum_{i=1}^{n} a_i\sqrt{x_i h_i}$, and solve this as an unconstrained problem.

Assessor's comment:

Another popular question that was well done by many undergraduates, though few postgraduates. The only systemic failing was an overlooking of the need to check second-order optimality conditions: only three candidates recognised the need to do this. As ever, a number of attempts were undermined by poor algebra or differentiation skills.