ENGINEERING TRIPOS PART IIB

Monday 3 May 2010 9 to 10.30

Module 4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A10 data sheet (two pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

mpj04

1 A simple model of a jet engine fuel injector consists of air of density ρ_1 flowing with uniform velocity U over a stationary liquid fuel of density ρ_2. The purpose of the air flow is to break up the liquid fuel into droplets.

(a) The fuel-air interface has surface tension σ. Gravity can be neglected. If the displacement of the fuel-air interface takes the form $\eta_0 \exp(st + \mathrm{i}kx)$, where t and x represent time and stream-wise distance respectively, show that:

$$s = -\mathrm{i}kU\frac{\rho_1}{\rho_1+\rho_2} \pm \left(\frac{k}{\rho_1+\rho_2}\right)^{\frac{1}{2}} \left(\frac{\rho_1\rho_2}{\rho_1+\rho_2}kU^2 - \sigma k^2\right)^{\frac{1}{2}}$$

[60 %]

(b) Find the most unstable wavenumber, k. [20 %]

(c) The Weber number is defined as:

$$\mathrm{We} \equiv \frac{\rho U^2 l}{\sigma}$$

where l is a characteristic length such as a droplet diameter or the thickness of a fluid sheet.

(i) What is the physical significance of the Weber number? [10%]

(ii) If the fuel were also moving with uniform velocity, what other influential non-dimensional group would also be important? [10%]

2 (a) A Couette flow is generated between an inner cylinder of radius R_1 , which is rotating with angular velocity Ω_1 , and an outer stationary cylinder of radius R_2 .

(i) Using energy arguments, derive Rayleigh's criterion for stability of an inviscid incompressible flow with circular streamlines. [30%]

(ii) Describe the physical mechanism that leads to instability in Couette flow and the resulting flow patterns as Ω_1 changes. [20%]

(b) A liquid is placed in a container and heated from below. The top surface of the liquid is open to the atmosphere.

(i) Give a physical description of the two types of instability that can occur as the heat input increases. [20%]

(ii) What non-dimensional groups are relevant to these instabilities and what is their significance? [20%]

(iii) When will one mechanism dominate over the other? [10%]

3 Two cylinders, one with square cross-section and one with circular cross-section, are suspended by springs between two fixed plates. This is modelled by the system shown in Fig. 1, in which all springs have the same spring constant, k, and both cylinders have mass m, length l into the page and side length or diameter d. The displacements of the masses from their equilibrium positions are y_1 and y_2. For the square cylinder, the change in the vertical force coefficient, C_y, with angle of attack, α, is shown in Fig. 2.

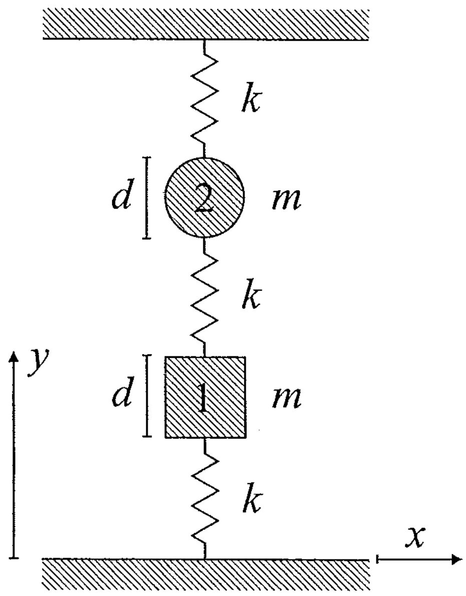

Fig. 1

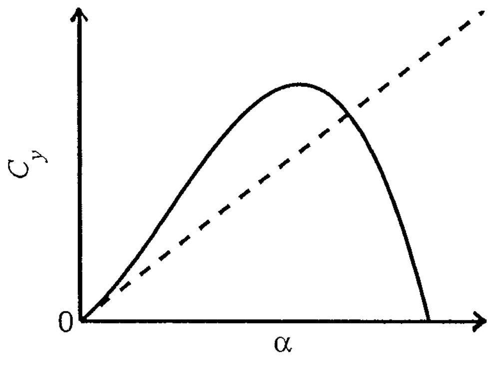

Fig. 2

(a) Find the resonant frequencies and corresponding mode shapes when the system has negligible damping and is placed in a vacuum. [20%]

(b) Repeat this analysis for the case where the damping is modelled by forces $-b\dot{y}_1$ on cylinder 1 and $-b\dot{y}_2$ on cylinder 2. Comment on your answer. [10%]

(c) The system is placed in a stationary fluid with density ρ . How does the fluid affect the resonant frequencies calculated in part (b)? Detailed calculations are not required. Estimate the density below which the resonant frequencies are affected by less than 1%. [20%]

(d) Assume that the fluid density is low enough not to affect the resonant frequencies of the system. The cylinders are initially at rest. The fluid moves in the x-direction at speed U , which increases slowly. Derive an expression for the critical speed at which the system starts to gallop. [20%]

(e) The fluid is moving at just below the critical speed calculated in part (d), at which the Reynolds number is approximately 10^4 . Describe the different types of vertical motion that are possible. Estimate the value or values of d at which the system will be particularly unstable. [30%]

4 A dish-shaped work of modern art has a mass, m , of 100 kg . When placed on a structure with stiffness k , the equation of motion perpendicular to the plane of the dish is:

$$m\ddot{x} + 2m\zeta\omega_n\dot{x} + kx = F$$

where the resonant frequency in the absence of damping, $\omega_n/(2\pi)$, is 2 Hz and the damping coefficient, ζ , is 0.3 .

(a) If the dish is forced at angular frequency ω and amplitude F_0 , what is the amplitude, x_0 , of the response at angular frequency ω ? [10%]

(b) The mechanical admittance, $|H_{Fx}(\omega)|^2$, is defined as the square of the transfer function between the forcing amplitude and the response amplitude. The transfer function is given by $|x_0/F_0|$. Find an expression for the mechanical admittance of the dish in terms of m , ω_n , ζ and ω . [10%]

(c) There is a strong wind perpendicular to the plane of the dish with mean velocity 30 ms^{-1} and a root-mean-square value of the fluctuations equal to 15% of the mean velocity. The wind fluctuation spectrum is constant for 0 to 4 Hz and zero for higher frequencies. Sketch the wind fluctuation spectrum, including the amplitude. [20%]

(d) The dish has a cross-sectional area, A , of 5 m^2 . The force on the dish is given by:

$$F = C_D \frac{1}{2} \rho U^2 A$$

where $C_D = 1$, $\rho = 1.25\ \text{kgm}^{-3}$ and U is the instantaneous velocity. Determine the aerodynamic admittance, $|H_{uF}(\omega)|^2$, of the structure. [20%]

(e) Sketch the spectrum of the deflection of the structure, indicating its maximum value. What would be the effect on the spectrum of the deflection if the natural frequency were increased, keeping the damping coefficient and the mass constant? [40%]

END OF PAPER

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EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

Continuity $\nabla \cdot \boldsymbol{u} = 0$

Navier Stokes $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

D/Dt denotes the material derivative, $\partial/\partial t + \boldsymbol{u}\cdot\nabla$

IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

velocity potential ϕ,

$$\boldsymbol{u} = \nabla\phi \quad \text{and} \quad \nabla^2\phi = 0$$

Bernoulli's equation

for inviscid flow $\dfrac{p}{\rho} + \tfrac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial\phi}{\partial t} = \text{constant throughout flow field.}$

KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u,v,w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + \boldsymbol{u}\cdot\nabla\eta \quad \text{on } z = \eta(x,t).$$

For η small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} \quad \text{on } z = 0.$$

SURFACE TENSION σ AT A LIQUID-AIR INTERFACE

Potential energy

The potential energy of a surface of area A is σA.

Pressure difference

The difference in pressure Δp across a liquid-air surface with principal radii of curvature R_1 and R_2 is

$$\Delta p = \sigma\left(\frac{1}{R_1} + \frac{1}{R_2}\right).$$

For a surface which is almost a circular cylinder with axis in the x-direction, $r = a + \eta(x,\theta,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma\left(-\frac{\eta}{a^2} - \frac{\partial^2\eta}{\partial x^2} - \frac{1}{a^2}\frac{\partial^2\eta}{\partial\theta^2}\right),$$

where Δp is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ (η is very small so that η^2 is negligible)

$$\Delta p = -\sigma\frac{\partial^2\eta}{\partial x^2}$$

where Δp is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius r only:

Rayleigh's criterion

The flow is unstable / stable to inviscid axisymmetric disturbances if Γ^2 decreases / increases with r.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius r.

Navier Stokes equation simplifies to

$$0 = \mu\left(\frac{d^2V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2}\right)$$

$$-\rho\frac{V^2}{r} = -\frac{dp}{dr}.$$

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STABILITY OF PARALLEL SHEAR FLOW

Rayleigh's inflexion point theorem

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

CONVECTIVE FLOW

The Boussinesq approximation leads to

$$\nabla\cdot\boldsymbol{u}=0$$

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))\boldsymbol{g}+\upsilon\nabla^2\boldsymbol{u}$$

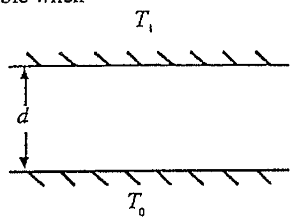
and

$$\frac{DT}{Dt}=\kappa\nabla^2 T$$

Rayleigh-Bénard convection

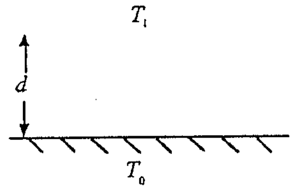
A fluid between two **rigid** plates is unstable when

$$\text{Ra}\geq 1708$$



A liquid with a **free** upper surface is unstable when

$$\frac{\text{Ra}}{\text{Ra}_c}+\frac{\text{Ma}}{\text{Ma}_c}\geq 1$$



where

$$\text{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \text{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa}\quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\text{Ra}_c\approx 670$ $\quad$ $\text{Ma}_c\approx 80.$

USEFUL MATHEMATICAL FORMULA

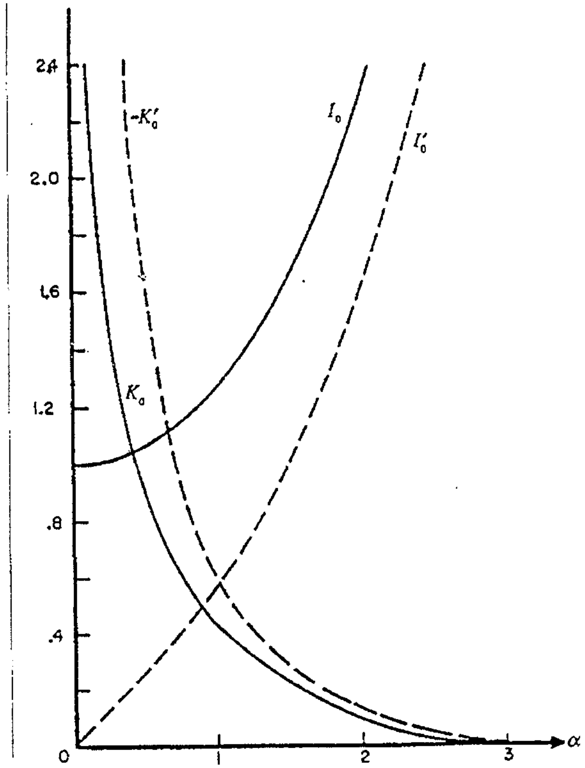
Modified Bessel equation

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.



$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative

 APD September 1998

4A10, 2010, Answers

Q1
(a) –
(b) $k = (U^2\rho_1\rho_2) / (4\sigma(\rho_1 + \rho_2))$
(c) –

Q2 –

Q3
(a) $\omega^2 = k/m$, masses moving in same direction; $\omega^2 = 3k/m$, masses moving in opposite direction.
(b) –
(c) fluid density less than 2% of the solid density.
(d) –
(e) $d = 2\pi U \, \mathrm{St}/\omega$

Q4
(a) –
(b) –
(c) $\max(S_{uu}) = 0.8057\ \mathrm{m^2s^{-1}}$
(d) $|H_{UF}|^2 = 35160\ \mathrm{N^2m^{-1}s^2}$
(e) $\max(S_{xx}) = 0.00034787\ \mathrm{m^2s}$