ENGINEERING TRIPOS PART IIB

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Friday 30 April 2010 9 to 10.30

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Module 4A12

# TURBULENCE AND VORTEX DYNAMICS

*Answer not more than* ***three*** *questions.*

*All questions carry the same number of marks.*

*The* ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachments:*
*4A12 Data Card: (i) Vortex Dynamics (1 page); (ii) Turbulence (2 pages)*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

PAD04

1 (a) Sketch the flow pattern for both Karman and Bodewadt boundary layers, showing the flow in the $r-\theta$ and $r-z$ planes. Why is there a radial component of motion in these two types of flow? [30%]

(b) Use order-of-magnitude analysis to show that, in both Karman and Bodewadt layers:

$$\rho\frac{u_\theta^2}{r} \sim \rho\nu\frac{u_r}{\delta^2}$$

where $\nu$ is the viscosity and $\delta$ is the boundary layer thickness. Hence show that $\delta$ is of the order of $\sqrt{\nu/\Omega}$, where $\Omega$ is the relative rate of rotation between the fluid and the solid boundary. [35%]

(c) Outside a Bodewadt layer the axial velocity is $u_z = 1.4\sqrt{\nu\Omega}$, where $\Omega$ is the rotation rate of the fluid relative to the stationary surface, and $z$ is measured from the surface. The equivalent result for a Karman layer is $u_z = -0.9\sqrt{\nu\Omega}$, where $\Omega$ is now the rotation rate of the surface relative to the distant fluid. A viscous fluid sits between two large, parallel, concentric disks. The top disk rotates at the rate $\Omega_0$ and the lower disk is stationary. Show that the bulk of the fluid between the disks rotates at $\Omega_f = 0.29\Omega_0$. [35%]

2 Consider the vortex sheet:

$$\boldsymbol{\omega} = \frac{\Phi}{\sqrt{\pi}\,\delta}\exp\left(-\frac{z^2}{\delta^2}\right)\hat{\mathbf{e}}_y$$

where $\Phi$ is a constant, $z$ is the axial coordinate, $\hat{\mathbf{e}}_y$ is a unit vector in the $y$-direction and $\delta$ is the characteristic thickness of the sheet. The velocity field associated with this vorticity distribution is $\mathbf{u}^{(\omega)} = u_x(z)\hat{\mathbf{e}}_x$.

(a) Sketch the vortex sheet and the associated velocity field, $\mathbf{u}^{(\omega)}$. [10%]

(b) Show that $\Phi$ is the flux of vorticity along the sheet (per unit length of the $x$-direction) and equal to the velocity jump across the sheet. You will need to use the fact that:

$$\int_{-\infty}^{\infty}\exp[-s^2]ds = \sqrt{\pi}$$ [20%]

(c) Show that $(\boldsymbol{\omega}\cdot\nabla)\mathbf{u}^{(\omega)} = 0$ and $(\mathbf{u}^{(\omega)}\cdot\nabla)\boldsymbol{\omega} = 0$. [10%]

(d) The vortex sheet is placed in an externally imposed, irrotational strain field:

$$\mathbf{u}^{(s)} = \left(u_x^{(s)}, u_y^{(s)}, u_z^{(s)}\right) = \left(0, \alpha y, -\alpha z\right)$$

where $\alpha$ is a constant and the total velocity field is now $\mathbf{u} = \mathbf{u}^{(s)} + \mathbf{u}^{(\omega)}$. Sketch $\mathbf{u}^{(s)}$ and show that, if $\delta = \sqrt{2\nu/\alpha}$, where $\nu$ is the viscosity, this constitutes an exact solution of the steady vorticity equation:

$$(\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + \nu\nabla^2\boldsymbol{\omega}$$ [40%]

(e) What physical processes are occurring in (d) which allows the vortex sheet to maintain a steady configuration? [20%]



(TURN OVER

3 A horizontal river of depth $h_1 = 1$ m and width $W_1 = 10$ m carries water at a uniform velocity of $U_1 = 1$ m/s and with negligible turbulence. It then flows down a steep waterfall of vertical height $H = 5$ m and resumes a horizontal path with depth $h_2 = h_1$ and width $W_2 = 5$ m.

(a) Estimate the turbulent velocity at the bottom of the waterfall. [30%]

(b) Make a reasonable estimate for the turbulent lengthscale at the bottom of the waterfall and then estimate the turbulent Reynolds number, the dissipation rate and the Kolmogorov lengthscale. [40%]

(c) State Taylor's hypothesis and discuss its validity in various part of the flow (before the waterfall, immediately after the waterfall and further downstream). [30%]

4 (a) Discuss briefly what is meant by the term *self-preservation* as applied to a turbulent round jet. Include comments on the structure of the turbulence and the relationship between various scaling quantities. [50%]

(b) The self-preserved turbulent round jet has a characteristic width $\delta$ that scales as $\delta \sim x$, where $x$ is the streamwise distance from the origin. Show that the scalar dissipation, $N$, along the centreline scales as $N \sim x^{-4}$. [50%]

**END OF PAPER**

# Vortex Dynamics Data Card

## Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left( \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left( \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right)$$

## Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot dS$

Stokes : $\int (\nabla \times A) \cdot d\boldsymbol{S} = \oint \boldsymbol{A} \cdot d\boldsymbol{l}$

## Vector Identities

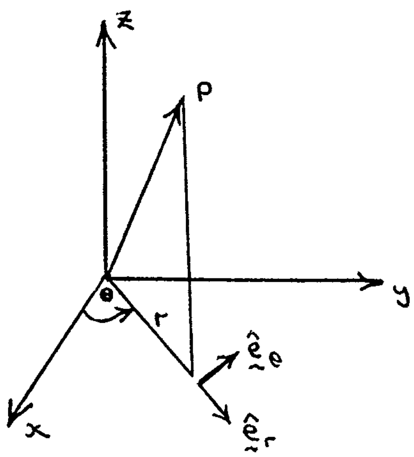
$\nabla(\boldsymbol{A} \cdot \boldsymbol{B}) = (\boldsymbol{A} \cdot \nabla)\boldsymbol{B} + (\boldsymbol{B}.\nabla)\boldsymbol{A} + \boldsymbol{A} \times (\nabla \times \boldsymbol{B}) + \boldsymbol{B} \times (\nabla \times \boldsymbol{A})$
$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot \nabla f$
$\nabla \times (\boldsymbol{A} \times \boldsymbol{B}) = \boldsymbol{A}(\nabla \cdot \boldsymbol{B}) - \boldsymbol{B}(\nabla \cdot \boldsymbol{A}) + (\boldsymbol{B} \cdot \nabla)\boldsymbol{A} - (\boldsymbol{A} \cdot \nabla)\boldsymbol{B}$
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times A) = 0$

## Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left( \frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z} \right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$




$$\nabla \times A = \left( \frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

# Cambridge University Engineering Department

## 4A12: Turbulence

## Data Card

Assume incompressible fluid with constant properties.

**Continuity:**

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

**Mean momentum:**

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

**Mean scalar:**

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

**Turbulent kinetic energy** ($k = \overline{u_i' u_i'}/2$):

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} - \overline{u_i' u_j'} \frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}$$

**Scalar fluctuations** ($\sigma^2 = \overline{\phi' \phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'} \frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

**Energy dissipation:**

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

**Scalar dissipation:**

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2$$

**Scaling rule for shear flow, flow dominant in direction $x_1$:**

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

**Kolmogorov scales:**

$$\begin{aligned} \eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\ \tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\ v_K &= \left(\nu\varepsilon\right)^{1/4} \end{aligned}$$

**Taylor microscale:**

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

**Eddy viscosity (general):**

$$\begin{aligned} \overline{u_i'u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\ \overline{u_j'\phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j} \end{aligned}$$

**Eddy viscosity (for simple shear):**

$$\overline{u_1'u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$

# ENGINEERING TRIPOS PART IIB 2010

## MODULE 4A12: TURBULENCE AND VORTEX DYNAMICS.

## NUMERICAL ANSWERS

Q1. –
Q2. –
Q3. (a) 5.63m/s, (b) Re = $5.6 \times 10^6$, $\varepsilon = 179$ $m^2 s^{-3}$, $\eta = 9$ $\mu$m, (c) -
Q4. -