ENGINEERING TRIPOS PART IIB

Wednesday 5 May 2010 2.30 to 4

Module 4C6

ADVANCED LINEAR VIBRATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4C6 Advanced Linear Vibration data sheet (10 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

JW05

1 A pressure vessel in a nuclear power plant is the subject of vibration modal analysis. An accelerometer is fixed to the wall of the vessel and an impulse is applied to two grid points in turn. Transfer functions are derived from the impulse response data and these are shown in Fig. 1 both as magnitude plots and as modal circles. For each measurement the data logger collects 8192 data points per channel at a sampling rate of 320 Hz. Three modes are identified at frequencies 25, 55 and 65 Hz.

(a) On a sketch, identify the frequency corresponding to each of the five visible modal circles. [10%]

(b) For each mode n, using the notation of the Data Sheet:

(i) estimate the quality factor Q_n; [25%]

(ii) estimate the modal amplitude factor $u_n(x)u_n(y)$ for each grid point. [20%]

(c) Use sketches to illustrate how the modal circles might appear if:

(i) the modal frequencies were 25, 64.5 and 65 Hz; [10%]

(ii) the sampling rate was increased to 3200 Hz. [10%]

(d) Comment on the signal-to-noise ratio revealed in these measurements. What factors might influence this? What steps might be taken to improve the ratio? [25%]

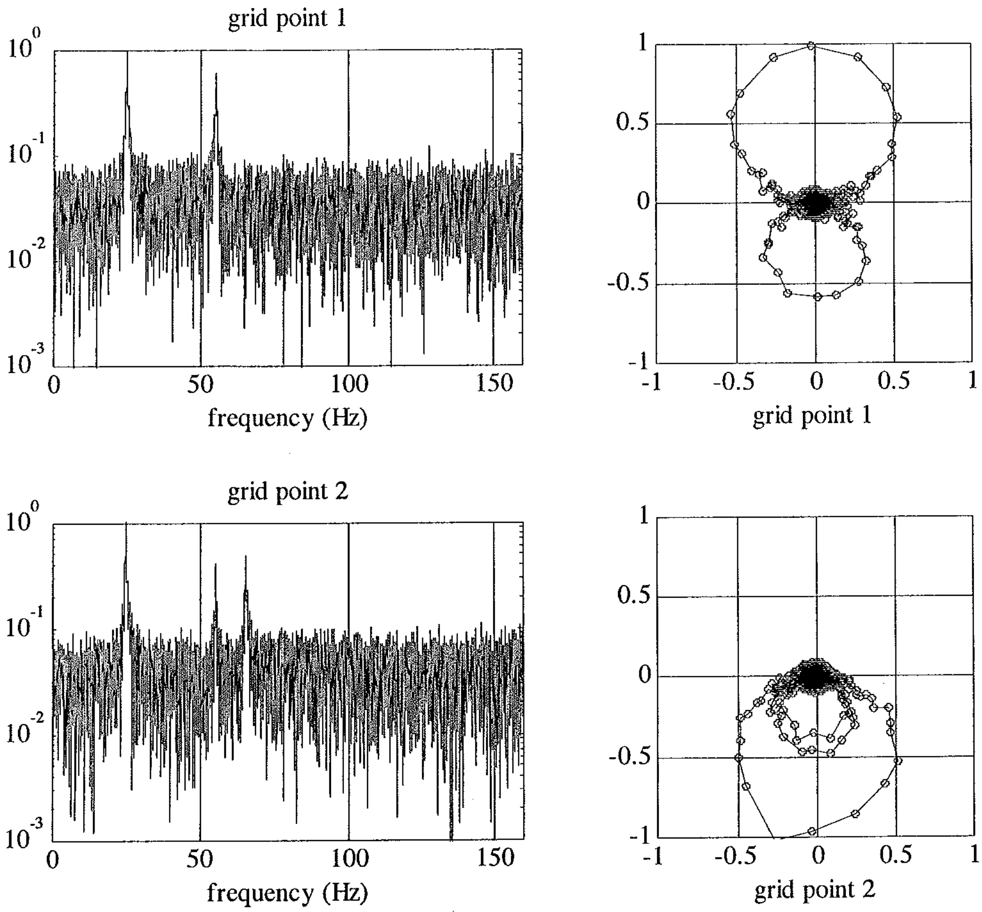

grid point 1
10^0
10^-1
10^-2
10^-3
0
50
100
150
frequency (Hz)
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
grid point 1
grid point 2
10^0
10^-1
10^-2
10^-3
0
50
100
150
frequency (Hz)
1
0.5
0
-0.5
-1
-1
-0.5
0
0.5
1
grid point 2

Fig. 1

2 (a) Briefly describe three different applications of the interlacing theorem. [25%]

(b) An H-shaped antenna and its mounting are represented schematically in Fig. 2. The four elements AC, AD, BE, BF are identical flexible beams. The member AB is a rigid link which clamps the flexible elements at their attachment points A and B. The antenna can vibrate laterally in the plane of the diagram so that the flexible elements execute small-amplitude bending motion. The supporting structure allows the link AB to move laterally, restrained by a spring.

(i) Explain how considerations of symmetry can be applied to the vibration modes of the antenna. Describe the vibration modes qualitatively and illustrate with appropriate sketches. [25%]

(ii) For the case in which the restraining spring is infinitely stiff, describe the distribution of natural frequencies. [15%]

(iii) The restraining spring is now reduced to a finite stiffness. What does the interlacing theorem say about which of the natural frequencies are unchanged and which may change? Explain the result physically in terms of the mode descriptions from part (i). [35%]

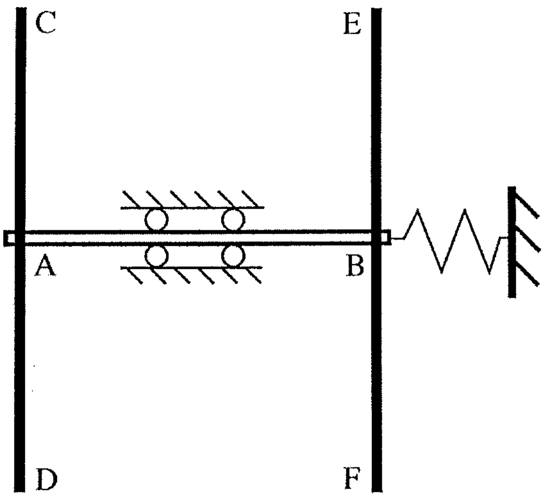


Fig. 2

3 For each of the following systems, describe the physical mechanisms which are likely to contribute significantly to vibrational energy dissipation. In each case explain how (if at all) the design and/or fabrication detailing could be changed (i) to introduce extra dissipation, and (ii) to reduce dissipation.

(a) A Meccano model of a structure (in other words, a model made of bolted metal components). [35%]

(b) A resonant MEMS device (a micro-scale mechanical oscillator made using silicon-chip fabrication methods, such as an air-bag accelerometer for a vehicle). [35%]

(c) The bounce and pitch modes of a railway carriage. [30%]

4 (a) A stretched rectangular membrane with dimensions $a \times b$, tension T and mass per unit area m is clamped on all sides. The out-of-plane displacement during small-amplitude vibration of the membrane is $w(x,y,t)$. Use the method of separation of variables to show that the mode shapes take the form $w = \sin k_1 x \sin k_2 y$ with particular values of k_1, k_2 which should be found. Hence find the natural frequencies of the membrane. [45%]

(b) Draw a sketch of the (k_1,k_2) plane showing the distribution of the modes and add to this sketch lines showing contours of equal natural frequency. [20%]

(c) For a frequency ω which corresponds to modes which are high up in the modal series for the membrane, use the sketch from part (b) to show that the approximate number of natural frequencies below ω can be expressed in terms of an integral over a region of the (k_1,k_2) plane. Deduce a formula for the *modal density* of the membrane (the approximate number of natural frequencies in a unit frequency interval), and show that this depends only on the area of the membrane and not on the aspect ratio a:b. [35%]

END OF PAPER

Part IIB Data sheet

Module 4C6 Advanced linear vibration

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M \underline{u}^{(n)}\ .$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 4 for examples.

$$T = \frac{1}{2}\int \dot{u}^2 dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 6) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\, \underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_j q_j(t)\, u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f} \;.$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2\, q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\,\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \geq 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \geq 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \geq 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \geq 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \geq 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 6.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\geq$ the smallest squared frequency;

(2) $\leq$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

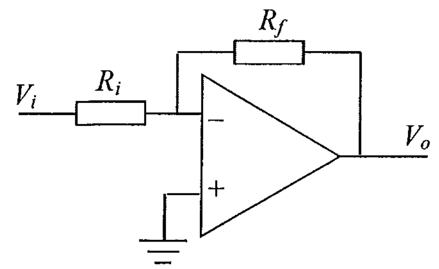
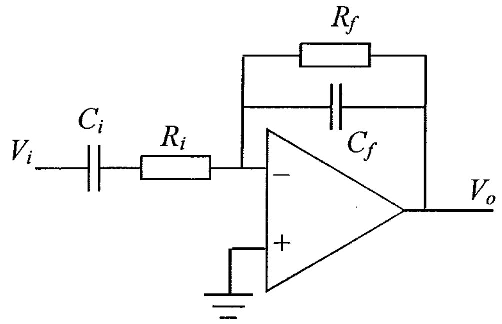
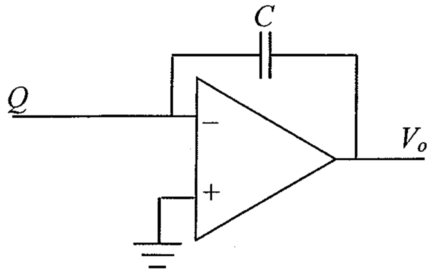
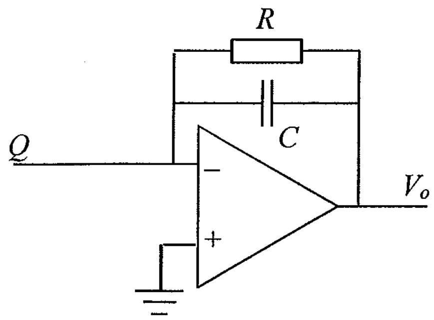
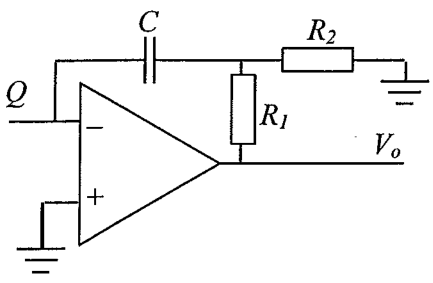
Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

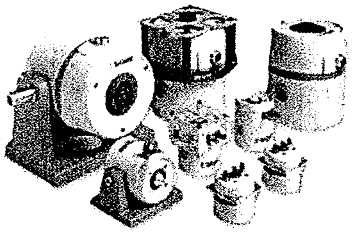
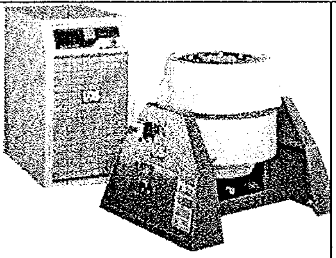
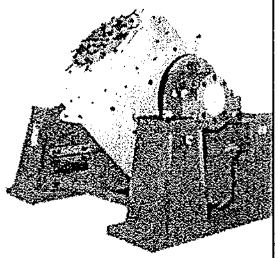
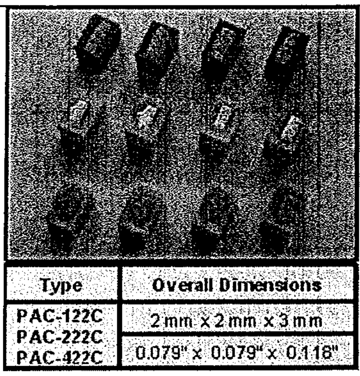
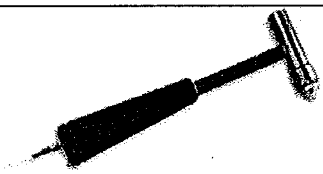
VIBRATION MEASUREMENT

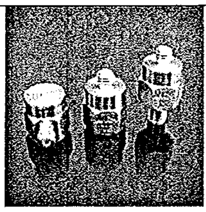
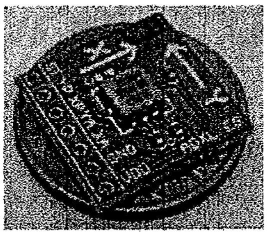
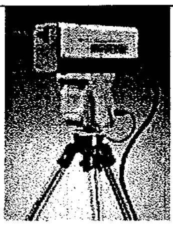
Some useful OpAmp circuits for instrumentation

(Note: j is used instead of i here for $\sqrt{-1}$ for compatibility with the Electrical Data Book.)

	Inverting voltage amplifier $V_o = -\frac{R_f}{R_i} V_i$
	Inverting voltage amplifier with low-pass and high-pass filter $V_o = -\frac{R_f}{R_i} \frac{V_i}{(1 + \frac{1}{j\omega R_i C_i})(1 + j\omega R_f C_f)}$
	Inverting charge amplifier $V_o = -\frac{Q}{C}$
	Inverting charge amplifier with high-pass filter $V_o = -\frac{Q}{C} \frac{1}{1 + \frac{1}{j\omega RC}}$
	Inverting charge amplifier with additional gain $V_o = -\frac{Q}{C} \frac{R_1 + R_2}{R_2}$

Some devices for vibration excitation and measurement

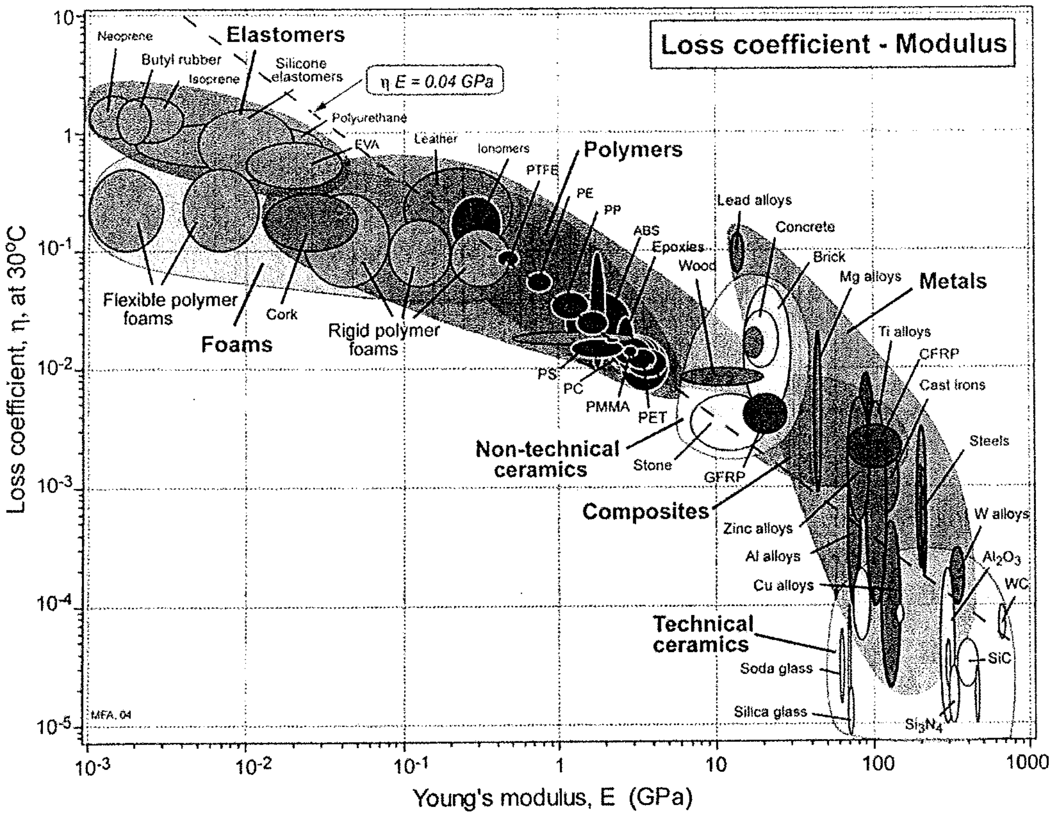
Moving coi electro-magnetic shaker	 LDS V101: Peak sine force 10N, internal armature resonance 12kHz Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, strok 2.5mm, shaker body mas 0.9kg	 LDS V650: Peak sine forc 1kN, internal armatur resonance 4kHz Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg	 LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz Frequency range 5 – 1.7kHz, armature suspension stiffne 72kN/m, armatur mass 250kg, strok 50mm, shaker bo mass 13000kg
Piezo stack actuator		FACE PAC-122C Size 2×2×3mm Mass 0.1g Peak force 12N Stroke 1μm Unloaded resonance 400kH	
Impulse hammer		IH101 Head mass 0.1kg hammer tip stiffness 1500kN/m Force transducer sensitivit 4pC/N Internal resonance 50kHz	

Piezo accelero-meter		B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale rang +/-5000g DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale rang +/-2000g B&K4370 Mass 10g sensitivity 100pC/g, 1-4.8kHz, , full-scale range +/-2000g	
MEMS accelero-meter		ADKL202E 265mV/g Full scale range +/- 2g DC-6kHz	
Laser Doppler Vibrometer		Polytec PSV-400 Scanning Vibrometer Velocity ranges 2/10/50/100/1000 [mm/s/V]	

VIBRATION DAMPING

Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \rightarrow E(1+i\eta)$. Typical values of E and η for engineering materials are shown below:



Free and constrained layers

For a 2-layer beam: if layer j has Young's modulus E_j, second moment of area I_j and thickness h_j, the effective bending rigidity EI is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh} \right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1}.$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12} \left[\frac{h_{31} - d}{1+g} \right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2} (h_{21} - d) + E_3 h_3 (h_{31} - d) \right] \left[\frac{h_{31} - d}{1+g} \right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

G_2 is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2} \underline{\dot{y}}^t C \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and C is the (symmetric) dissipation matrix.

If the system has mass matrix M and stiffness matrix K, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = 0.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}$, λ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and I is the unit matrix.

THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume V with a neck of effective length L and cross-sectional area S has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound in air.

The end correction for an unflanged circular neck of radius a is $0.6a$.

The end correction for a flanged circular neck of radius a is $0.8a$.

VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension T and mass per unit area m undergoes small transverse free vibration with displacement w, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates x, y or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates r,θ.

For a circular membrane of radius a the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where J_n is the Bessel function of order n and k is determined by the condition that $J_n(ka) = 0$. The first few zeros of J_n's are as follows:

	$n=0$	$n=1$	$n=2$	$n=3$
$ka =$	2.404	3.832	5.135	6.379
$ka =$	5.520	7.016	8.417	9.760
$ka =$	8.654	10.173		

For a given k the corresponding natural frequency ω satisfies

$$k = \omega\sqrt{m/T}.$$