ENGINEERING TRIPOS PART IIB

Wednesday 21 April 2010 2.30 to 4

Module 4C9

CONTINUUM MECHANICS

Answer not more than **two** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C9 datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

jaw03

1 (a) (i) If $A_{ij} = \delta_{ij} B_{kk} + 3B_{ij}$ determine B_{kk} and using this result solve for B_{ij} in terms of A_{ij} and its first invariant A_{ii}. [25%]

(ii) For the position vector x_i having magnitude x, show that $x_{,j} = \dfrac{x_j}{x}$ [25%]

(b) (i) Explain briefly the significance of Druker's postulates for a von Mises elastic-plastic solid. Your answer should include reference to the concepts of stability, convexity and normality. [30%]

(ii) A thin-walled circular tube of length ℓ, radius r and wall thickness t is made from an elastic, ideally plastic solid of Young's Modulus E and uniaxial yield strength σ_Y. It is to be loaded in pure torsion.

Determine the torque and twist at first yield. [10%]

Sketch the torque versus twist response beyond first yield. [10%]

2 Figure 1 shows an apparatus for the plane strain "back extrusion" of a slab in which the thickness is reduced from $6b$ to $2b$ by a pair of rigid dies. The material being extruded has a yield stress in pure shear k.

(a) Use the upper-bound method, with lines of tangential velocity discontinuity as shown dotted, with x variable, to make an estimate of the minimum total force required per unit width to perform the extrusion. Assume that friction is negligible. [50%]

(b) Discuss briefly how the estimate would be modified if friction forces acted on the sloping faces of the dies. [25%]

(c) Explain briefly why the geometry of this solution is not a valid slip line field. [25%]

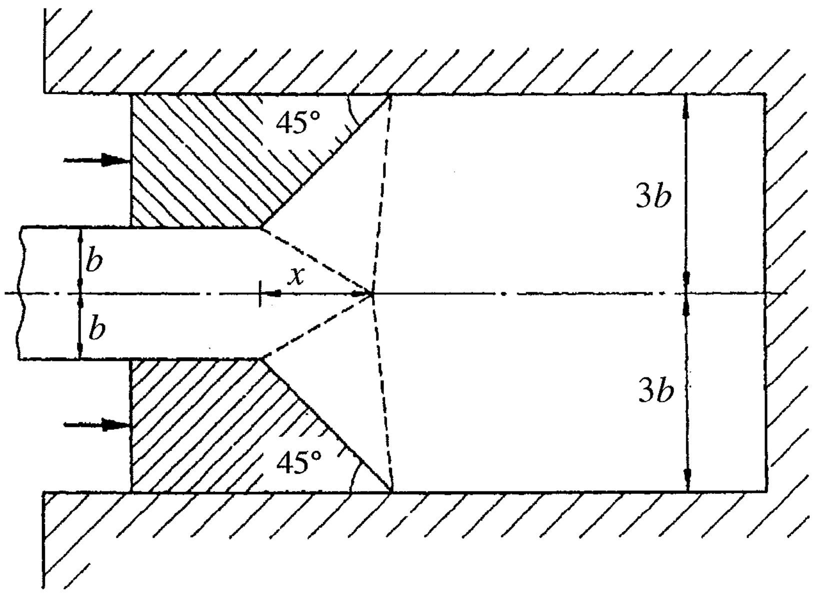

Fig. 1

3 (a) Figure 2(a) shows part of an infinite continuous 2-D body within which a point P can be located by either the Cartesian coordinates (x,y) or the polars (r,θ). By considering the stress function

$$\phi = -Dr\ln r\cos\theta$$

and using information from the data sheet, write down expressions for the elastic stress components σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ at P in terms of its polar coordinates. [10%]

(b) By making use of the rotation matrix which relates the Cartesian axes to the polar set, obtain expressions for the values of the stress components σ_{xx}, σ_{yy} and σ_{xy} at P in terms of its Cartesian co-ordinates. Simplify these expressions for elements which lie along the along the positive $x-$axis. Sketch their variation for $0 \le x \le a$. [30%]

(c) The specified stress function is to be used to investigate the elastic stresses associated with a dislocation which can be modelled by imagining a cut along the positive $x-$axis with equal and opposite tractions applied to the faces so formed so that a gap of constant thickness δ is opened between them, as illustrated in Fig. 2(b). The strength of the dislocation can be described by the magnitude of the discontinuity δ in the displacement u_θ on the planes $\theta = 0$ and $\theta = 2\pi$. By assuming plane stress and making use of the data sheet, show that

$$\delta = \frac{2\pi D}{G(1+\nu)}$$

where G is the shear modulus and ν the Poisson's ratio of the material. [30%]

(d) This solution is to be used to analyse the stresses in the vicinity of a planar crack within a tensile field as illustrated in Fig. 2(c). The full solution can be taken to be the sum of the stress field in the uncracked body, i.e. $\sigma_{yy} = S$ and $\sigma_{xx} = \sigma_{xy} = 0$ everywhere, together with a second solution for which the boundary conditions are

$$\sigma_{xy} = 0, \quad \sigma_{yy} = -S \quad \text{for} \quad -a < x < +a \quad \text{and} \quad y = 0$$
$$\text{and} \quad \sigma_{xx}, \ \sigma_{xy} \ \text{and} \ \sigma_{yy} \to 0 \quad \text{for} \quad r \to \infty$$

By considering the traction σ_{yy} at the point $(x,0)$ due to the dislocations located between $(\xi,0)$ and $(\xi+\mathrm{d}\xi,0)$, show that in this second solution

$$\sigma_{yy} = -\frac{G(1+\nu)}{2\pi}\int_{-a}^{+a}\frac{\delta(\xi)}{(x-\xi)}\,\mathrm{d}\xi$$

where $\delta(\xi)$ is the strength of the dislocations at the point $(\xi,0)$. [30%]

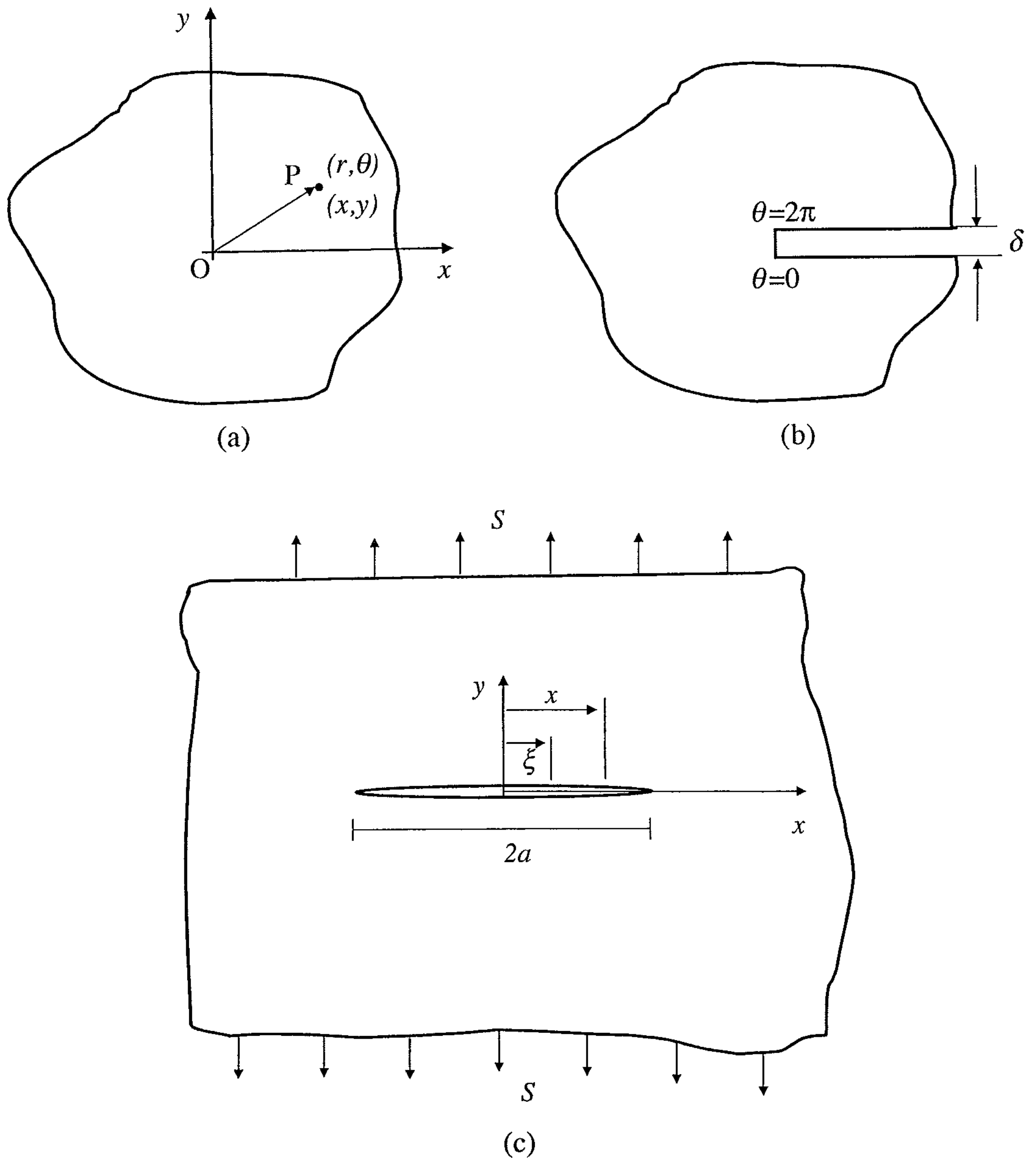

Fig. 2

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underline{a} = a_1 \underline{e}_1 + a_2 \underline{e}_2 + a_3 \underline{e}_3$	$a_i e_i$
$\underline{a} \bullet \underline{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underline{c} = \underline{a} \times \underline{b}$	$c_i = e_{ijk} a_j b_k$
$\underline{d} = \underline{a} \times (\underline{b} \times \underline{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	$e_{ijk} = 1$ when indices cyclic; $= -1$ when indices anticyclic and $= 0$ when any indices repeat
$e - \delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$
trace$a$	$\text{tr}a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\frac{\partial \sigma_{ij}}{\partial x_i} = \frac{\partial \sigma_{1j}}{\partial x_1} + \frac{\partial \sigma_{2j}}{\partial x_2} + \frac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
grad$\phi = \nabla \phi$	$\frac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underline{V}$	$V_{i,i}$
curl$\underline{V} \equiv \underline{\nabla} \times \underline{V}$	$e_{ijk} V_{k,j}$

<u>Rotation of Orthogonal Axes</u>

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes $\quad v'_\alpha = a_{\alpha i} v_i$

tensor σ_{ij} becomes $\quad \sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Dec 09

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$I_1 = \sigma_{ii} = \mathrm{tr}\sigma$$
$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$
$$I_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\,\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$$

$s^3 - J_2 s - J_3 = 0$

$$J_1 = s_{ii} = \mathrm{tr}s \; ; \; J_2 = \frac{1}{2}s_{ij}s_{ij} \; ; \; J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

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equilibrium $\qquad \sigma_{ij,i} + b_j = 0$

small strains

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

compatibility $\qquad \varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik}e_{qjl}\varepsilon_{ij,kl} = 0$

equivalent to $\quad e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\dfrac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $\qquad E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\qquad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

von Mises equivalent stress $\qquad \sigma_e \equiv \bar{\sigma} = \sqrt{\dfrac{3}{2}s_{ij}s_{ij}} = \sqrt{3J_2}$

equivalent strain increment $\qquad \mathrm{d}\bar{\varepsilon} = \sqrt{\dfrac{2}{3}\mathrm{d}\varepsilon_{ij}\mathrm{d}\varepsilon_{ij}}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\quad \nabla^2\Psi = \Psi_{,ii} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ii} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi_{,2} = \frac{\partial\phi}{\partial x_2} \;, \quad \sigma_{32} = -\phi_{,1} = -\frac{\partial\phi}{\partial x_1} \quad \text{and} \quad T = 2\iint_A \phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G=\mu$	λ
E, ν	–	–	$\frac{E}{2(1+\nu)}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\frac{E-2G}{2G}$	–	$\frac{(2G-E)G}{E-3G}$
E, λ	–	$\frac{E-\lambda+R}{4\lambda}$	$\frac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\frac{2G\nu}{1-2\nu}$
ν, λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\frac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda}{2(\lambda+G)}$	–	–

$R=\sqrt{E^2+2E\lambda+9\lambda^2}$

Two-dimensional Airy Stress function

Biharmonic equation $\nabla^4\phi \equiv \phi_{,\alpha\alpha\beta\beta}=0$

Stresses $\sigma_{\alpha\beta}=e_{\alpha\gamma}\, e_{\beta\delta}\, \phi_{,\gamma\delta}$

$$\text{where } e_{\alpha\beta} \equiv e_{3\alpha\beta} = \begin{cases} 1 \text{ if } \alpha=1, \beta=2 \\ 0 \text{ if } \alpha=\beta \\ -1 \text{ if } \alpha=2, \beta=1 \end{cases}$$

Plane stress and plane strain

$$G\varepsilon_{11}=\frac{1}{8}\{\sigma_{11}(1+\kappa)+\sigma_{22}(\kappa-3)\}$$

$$G\varepsilon_{22}=\frac{1}{8}\{\sigma_{22}(1+\kappa)+\sigma_{11}(\kappa-3)\}$$

$$G\varepsilon_{12}=\frac{\sigma_{12}}{2}$$

$$\text{where } \begin{cases} \kappa=(3-\nu)/(1+\nu) & \text{in plane stress and} \\ \kappa=3-4\nu & \text{in plane strain} \end{cases}$$

Plasticity

von Mises yield criterion $f = \sigma_e - Y = 0$

J_2 flow rule $\dot{\varepsilon}_{ij}^{PL} = \frac{3}{2} \frac{s_{ij}}{\sigma_e} \frac{\dot{\sigma}_e}{h}$

Slip Line Fields

Henky equations

$p + 2k\phi = \text{constant along } \alpha \text{ line}$

$p - 2k\phi = \text{constant along } \beta \text{ line}$

Geiringer equations

$\frac{\mathrm{d}v_\alpha}{\mathrm{d}s} = v_\beta \frac{\mathrm{d}\phi}{\mathrm{d}s}$ along α line

$\frac{\mathrm{d}v_\beta}{\mathrm{d}s} = -v_\alpha \frac{\mathrm{d}\phi}{\mathrm{d}s}$ along β line

Table I – The Michell solutions — stress components

$\phi(r,\theta)$	σ_{rr}	$\sigma_{\theta\theta}$	$\sigma_{r\theta}$
r^2	2	2	0
$r^2 \ln r$	$2\ln r + 1$	$2\ln r + 3$	0
$\ln r$	$1/r^2$	$-1/r^2$	0
θ	0	0	$1/r^2$
$r^3 \cos\theta$	$2r\cos\theta$	$6r\cos\theta$	$2r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln r\cos\theta$	$\cos\theta/r$	$\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$2\cos\theta/r^3$	$-2\sin\theta/r^3$
$r^3 \sin\theta$	$2r\sin\theta$	$6r\sin\theta$	$-2r\cos\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$\sin\theta/r$	$-\cos\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n \cos n\theta$	$(n+1)(n+2)r^n \cos n\theta$	$n(n+1)r^n \sin n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n} \cos n\theta$	$(n-1)(n-2)r^{-n} \cos n\theta$	$-n(n-1)r^{-n} \sin n\theta$
$r^n \cos n\theta$	$-n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \sin n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2} \cos n\theta$	$n(n+1)r^{-n-2} \cos n\theta$	$-n(n+1)r^{-n-2} \sin n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n \sin n\theta$	$(n+1)(n+2)r^n \sin n\theta$	$-n(n+1)r^n \cos n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n} \sin n\theta$	$(n-1)(n-2)r^{-n} \sin n\theta$	$n(n-1)r^{-n} \cos n\theta$
$r^n \sin n\theta$	$-n(n-1)r^{n-2} \sin n\theta$	$n(n-1)r^{n-2} \sin n\theta$	$-n(n-1)r^{n-2} \cos n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2} \sin n\theta$	$n(n+1)r^{-n-2} \sin n\theta$	$n(n+1)r^{-n-2} \cos n\theta$

Table II – The Michell solutions — displacement components

For plane strain $\kappa = 3 - 4\nu$; for plane stress $\kappa = (3-\nu)/(1+\nu)$

$\phi(r,\theta)$	$2Gu_r$	$2Gu_\theta$
r^2	$(\kappa-1)r$	0
$r^2 \ln r$	$(\kappa-1)r\ln r - r$	$(\kappa+1)r\theta$
$\ln r$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$0.5[(\kappa-1)\theta\sin\theta-\cos\theta + (\kappa+1)\ln r\cos\theta]$	$0.5[(\kappa-1)\theta\cos\theta-\sin\theta - (\kappa+1)\ln r\sin\theta]$
$r\ln r\cos\theta$	$0.5[(\kappa+1)\theta\sin\theta-\cos\theta + (\kappa-1)\ln r\cos\theta]$	$0.5[(\kappa+1)\theta\cos\theta-\sin\theta - (\kappa-1)\ln r\sin\theta]$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa-2)r^2\cos\theta$
$r\theta\cos\theta$	$0.5[(\kappa-1)\theta\cos\theta+\sin\theta - (\kappa+1)\ln r\sin\theta]$	$0.5[-(\kappa-1)\theta\sin\theta-\cos\theta - (\kappa+1)\ln r\cos\theta]$
$r\ln r\sin\theta$	$0.5[-(\kappa+1)\theta\cos\theta-\sin\theta + (\kappa-1)\ln r\sin\theta]$	$0.5[(\kappa+1)\theta\sin\theta+\cos\theta + (\kappa-1)\ln r\cos\theta]$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa+n-1)r^{-n+1}\cos n\theta$	$-(\kappa-n+1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa+n-1)r^{-n+1}\sin n\theta$	$(\kappa-n+1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

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Answers

1 (a) (i) $B_{ii} = \frac{1}{6}A_{ii} \quad B_{ij} = \frac{1}{3}A_{ij} - \frac{1}{18}\delta_{ij}A_{kk}$

(b) (ii) $\frac{2\pi r^2 t\sigma_Y}{\sqrt{3}} \quad \frac{\ell\sigma_Y}{\sqrt{3}Gr}$

2 (a) $15.2kb$ (b) for sticking friction $24kb$

3 (a) $\sigma_{rr} = -\frac{D\cos\theta}{r}, \quad \sigma_{\theta\theta} = -\frac{D\cos\theta}{r}, \quad \sigma_{r\theta} = -\frac{D\sin\theta}{r}$

(b) $\sigma_{xx} = -\frac{Dx\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}, \quad \sigma_{xx} = -\frac{Dy\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}, \quad \sigma_{xy} = \frac{Dy\left(x^2 - y^2\right)}{\left(x^2 + y^2\right)^2}$

$$\sigma_{xx} = -\frac{D}{x}, \quad \sigma_{yy} = -\frac{D}{x}, \quad \sigma_{xy} = 0$$