ENGINEERING TRIPOS PART IIB

Wednesday 21 April 2010 2.30 to 4

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Formulae sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed
Supplementary pages: Three extra copies of Fig. 2 (Question 2).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

mcs02

1 A position control system is to be designed for a tracking antenna in the arrangement of Fig. 1 where $r(t)$ is the reference input and $y(t)$ is the pointing angle. The open-loop transfer-function of the antenna from actuator input to pointing angle in arcmin is approximated as:

$$G(s) = \frac{1}{s(s+1)}.$$

(a) Consider the following specifications:

A: Steady-state error of at most 0.1 arcmin for $r(t)$ equal to a ramp of 10 arcmin s^{-1};

B: Phase margin of at least 45°.

Design a simple controller $K(s)$ with one pole and one zero to meet these specifications and to achieve an actual phase margin of between 50° and 55°. Give a rough sketch of the Bode diagram of the resulting return ratio to illustrate the approach. [40%]

(b) It is desired that the sensitivity of the control system is significantly reduced over as wide a frequency range as possible. Plant uncertainty at high frequency is also a concern. The following specifications are therefore investigated, where $S(s)$ denotes the sensitivity function:

C: $|S(j\omega)| < 0.1$ for $0 \leq \omega \leq \omega_1$;

D: $|S(j\omega)| < \sqrt{2}$ for all ω;

E: $|G(j\omega)K(j\omega)| \leq 100/\omega^2$ for $\omega \geq 100$ rad s^{-1}.

Find an upper bound on the achievable ω_1. State clearly but do not prove any results you use. [Hint: you may use the approximation: $\ln(1+x) \simeq x$ for x small.] [35%]

(c) Discuss how the compensator $K(s)$ of part (a) could be modified to achieve specifications A, B and E. It is no longer required that $K(s)$ has one pole and zero. It is not necessary to calculate the final compensator, but sufficient explanation is needed for your approach to be clear. [25%]

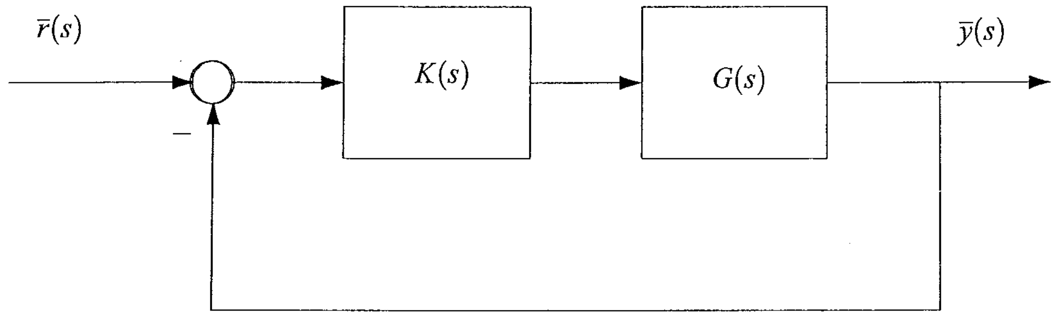


Fig. 1

2 Fig. 2 is the Bode diagram of a system $G(s)$ for which a feedback compensator $K(s)$ in the standard negative feedback configuration is to be designed. It is known that $G(s)$ has exactly one pole in the right half-plane.

(a) (i) Sketch on a copy of Fig. 2 the expected phase of $G(j\omega)$ if $G(s)$ were stable and minimum phase. [10%]

(ii) What does this plot suggest about possible zeros of $G(s)$ satisfying $\mathrm{Re}(s) \geq 0$? [15%]

(b) A double lead compensator with transfer function

$$K(s) = k\left(\frac{\alpha s + \omega_c}{s + \omega_c \alpha}\right)^2$$

where $\alpha > 1$ is selected.

(i) Let $k = 1$, $\omega_c = 1$, $\alpha = \sqrt{10}$. Sketch the Bode diagram of $K(s)$ and the resulting return ratio on a copy of Fig. 2. [10%]

(ii) Sketch the Nyquist diagram of the return ratio and hence explain why the closed-loop system can never be stable with this ω_c, α and any k. [20%]

(c) (i) Explain using the root-locus method why the closed-loop system can never be stable with a compensator $K(s)$ which has all its poles in the left half-plane. [15%]

(ii) The closed-loop system can be made stable with $K(s)$ equal to a constant times a p-type all-pass function. Select suitable parameters for such a $K(s)$ and sketch the Bode diagram of $K(s)$ and the resulting return ratio on a copy of Fig. 2. Justify that the closed-loop system is stable using the Nyquist stability criterion. [30%]

Three copies of Fig. 2 are provided on separate sheets. These should be handed in with your answers.

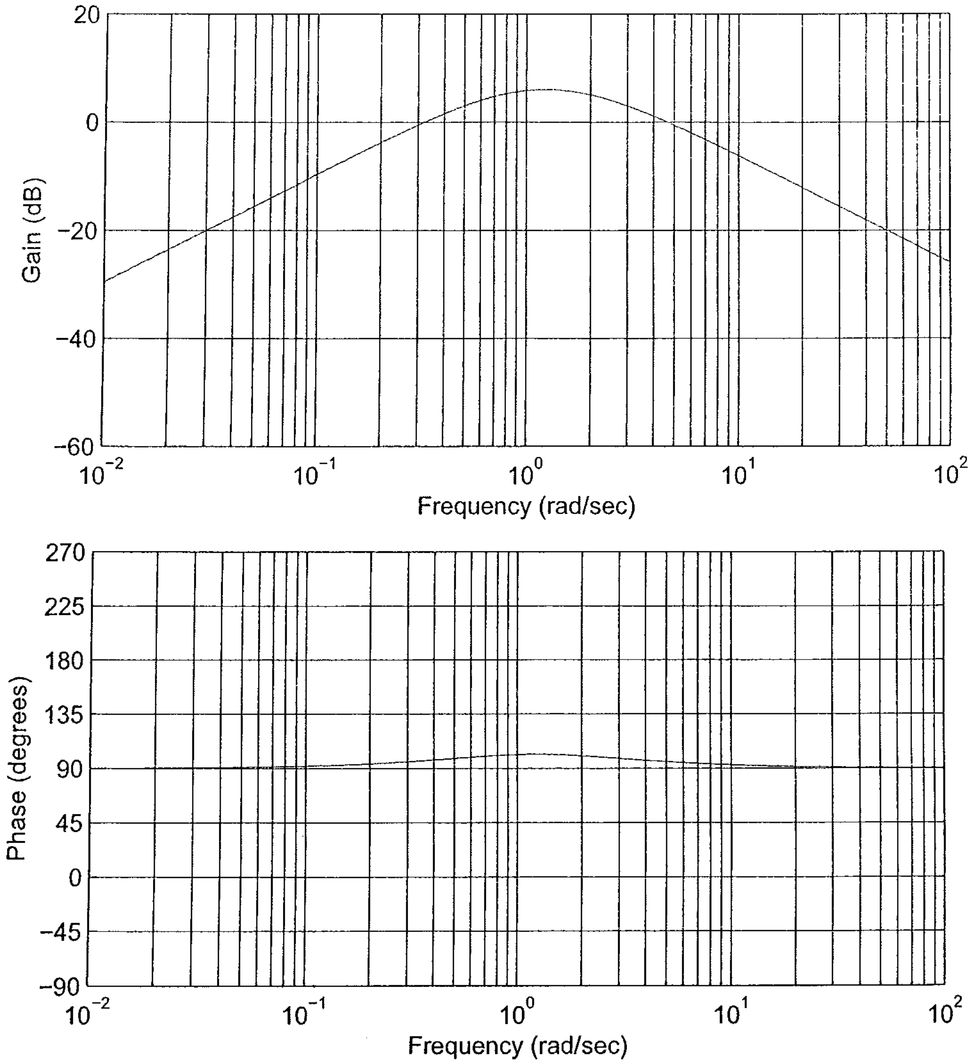

20
0
-20
-40
-60
Gain (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)
270
225
180
135
90
45
0
-45
-90
Phase (degrees)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
Frequency (rad/sec)

Fig. 2

3 (a) State but do not prove the conditions which must be satisfied by the transfer-function from reference input to plant output in any two-degree-of-freedom control system design. [10%]

(b) Consider a plant with transfer-function:

$$G(s) = \frac{\prod_{i=1}^{m}(1+\alpha_i s)}{\prod_{i=1}^{n}(1+\beta_i s)} \qquad (1)$$

where $\beta_i > 0$ for all i and $m < n$.

(i) Let $m = n-1$ and suppose that $\alpha_i \neq 0$ for all i. Find an expression for the initial slope of the step response of the plant. Hence show that there is initial undershoot in the step response if and only if an odd number of the α_i are negative. [15%]

(ii) Suppose that $\alpha_k < 0$ for some k. By considering the definition of the Laplace transform of the step response $y(t)$, or otherwise, show that [15%]

$$\int_0^\infty y(t)e^{t/\alpha_k}dt = 0.$$

(iii) Deduce that, if one or more of the α_i is negative, then there will always be at least one $t_1 > 0$ where $y(t_1) = 0$. [15%]

(c) Let

$$G(s) = \frac{(1-s)^2}{(1+s)^4}.$$

(i) Find the initial slope, and first derivative of the initial slope, of the step response. [15%]

(ii) Without attempting to calculate it, provide an approximate sketch of the step response of $G(s)$ which indicates the main features that may be expected. [15%]

(iii) If a two-degree-of-freedom control system is designed for this plant, what features of the step response are unavoidable, assuming that the unity DC gain of the response is retained? [15%]

END OF PAPER

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

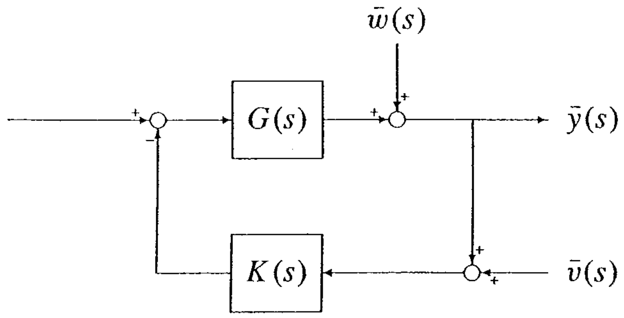
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \quad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90°.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\mathrm{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\,dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

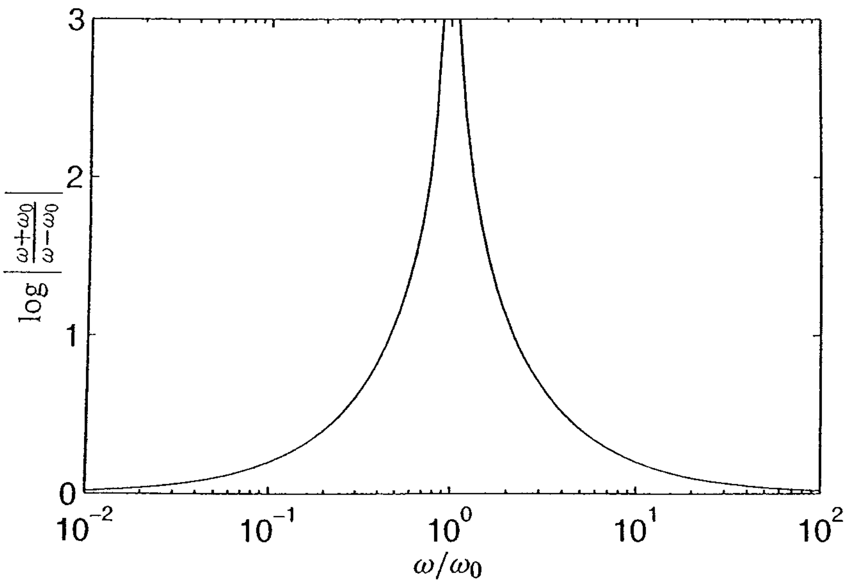


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2}\left.\frac{d\log|L(j\omega_0 e^v|)}{dv}\right|_{\omega=\omega_0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\text{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\left(j|s_0|e^{v}\right)\right| dv}$$

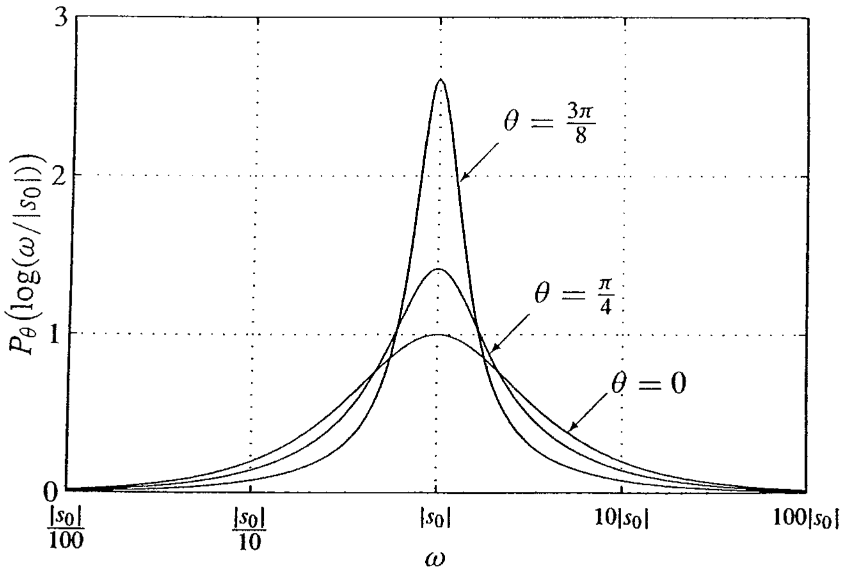
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

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November 2002

4F1 2010 — Answers

1(a) Spec. A implies $K(0) \geq 100$. (b) $\omega_1 \leq 13.49$ rad/sec.

2(a)(ii) $G(s)$ has a zero at $s = 0$ but no other zeros in the right half-plane.
(c)(ii) $K(s) = \dfrac{1+s}{1-s}$ is one possibility.

3(b)(i) Initial slope of step response:

$$\frac{\prod_{i=1}^{n-1} \alpha_i}{\prod_{i=1}^{n} \beta_i}$$

(c)(i) Initial slope of step response: 0. Derivative of initial slope of step response: 1.

M.C. Smith, 24 May 2010