ENGINEERING TRIPOS   PART IIB

Wednesday 28 April 2010   2.30 to 4

Module 4F2

ROBUST MULTIVARIABLE CONTROL

Answer not more than ***two*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

jmg03

1 Consider the feedback system in Fig. 1 and let $G(s)$ be the transfer function from w to u.

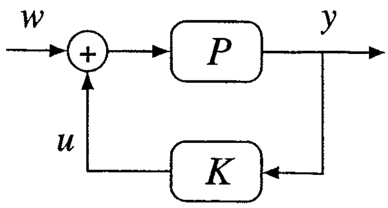


Fig. 1

(a) Write down the relationship between the $\mathscr{H}_\infty$ norm of $G(s)$, and the $\mathscr{L}_2$ norms of w and u. [10%]

(b) To estimate an upper bound on the value of $\|G(s)\|_\infty$, consider the quadratic Lyapunov function:

$$V(x) = x^T M x,$$

where M is a symmetric and positive definite matrix. Show that if $V(x)$ is a Lyapunov function and if

$$\frac{dV}{dt} \leq \gamma^2 w^T w - u^T u \qquad (1)$$

then

$$\frac{\|u\|_2}{\|w\|_2} \leq \gamma.$$

[25%]

(c) Suppose that the state of the plant P in Fig. 1 is fully observed (i.e. $y = x$), that P has the state space realization:

$$\frac{dx}{dt} = Ax + w + u$$

and that $u = -Mx$. Show that inequality (1) above can be satisfied by choosing the matrix M and the scalar γ such that the following three conditions hold:

C1 The matrix M and the scalar γ satisfy the Riccati equation:

$$MA + A^T M - M^2 + \frac{1}{\gamma^2} M^2 = 0$$

C2 The matrix $A - M$ has all its eigenvalues in the open left half-plane;

C3 The matrix $A - M + \frac{1}{\gamma^2}M$ has all its eigenvalues in the open left half-plane.

[40%]

(d) If the plant is the scalar system given by:

$$\frac{dx}{dt} = x + u + w,$$

show that conditions C1, C2 and C3 in part (c) imply that:

$$\gamma > 1.$$

[25%]

2 Consider the discrete-time controlled dynamical system given by:

$$x_{k+1} = f(x_k, u_k), \text{ for } k \in \{1, 2, \ldots, h-1\},$$

where h is a given positive integer. We seek controls $(u_1, u_2, \ldots, u_{h-1})$ that minimise the cost:

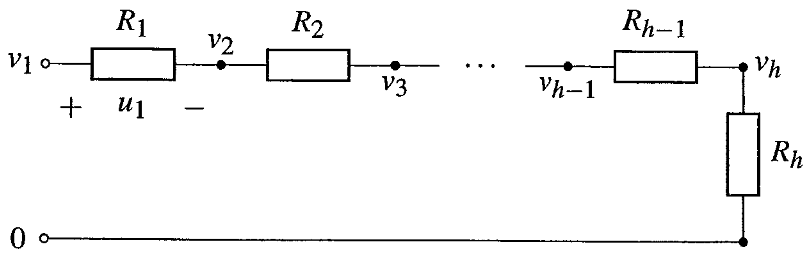
$$J = \sum_{k=1}^{h-1} c(x_k, u_k) + J_h(x_h),$$

where $c(\cdot,\cdot)$ and $J_h(\cdot)$ are non-negative functions.

(a) Define the *value function* $V(x,k)$ for this problem and explain the significance of $V(x,h)$ and $V(x,0)$. [25%]

(b) State the dynamic programming equation that $V(x,k)$ must satisfy. [15%]

(c) We now study the division of an applied voltage across a series combination of h ideal resistors. In particular, we seek the voltage drops across the individual resistors that minimise the total heat lost in all the resistors. We want to show that this result is the same as the one obtained by applying Ohm's and Kirchoff's laws.



If we treat the sequence of voltages

$$v_1, v_2, \ldots, v_h$$

as a sequence of state variables, and the voltage drop across a resistor R_k (for $k < h$) as the 'control' u_k, then we can write the state space equations as:

$$v_{k+1} = v_k - u_k, \text{ for } k = 1, 2, \ldots, h-1.$$

We want to minimise the total power lost as heat dissipated per unit time. It is given by:

$$J_{\text{Thermal}} = \sum_{k=1}^{h-1} \frac{u_k^2}{R_k} + \frac{v_h^2}{R_h},$$

(i) Write down the value function at 'time' h, and the dynamic programming equations for choosing u_k optimally for $k = 1, 2, \ldots, h-1$. [10%]

(ii) Show that quadratic value functions solve the dynamic programming equations. [30%]

[*Hint*: You may find the following result useful. Given Q, S and R, with $R > 0$, then

$$\min_u \left(x^2 Q + 2xuS + u^2 R\right) = x^2 \left(Q - \frac{S^2}{R}\right)$$

and the minimum is achieved at $u = -\frac{S}{R}x$.]

(iii) Find the optimal set of voltage drops taking the 'initial state' v_1 as a fixed parameter. Hence, show that the voltage drops across the individual resistors that minimise the total heat lost in all the resistors are equal to those obtained by applying Ohm's and Kirchoff's laws. [20%]

3 (a) State the *small gain theorem*, defining any terms used. [20%]

(b) Show, with the aid of a diagram, that a controller K stabilises all systems of the form

$$(I+\Delta W)^{-1}G, \; \|\Delta\|_\infty < \varepsilon$$

if, and only if,

$$\|W(I+GK)^{-1}\|_\infty \leq 1/\varepsilon.$$

W and Δ are taken to be stable transfer function matrices, and a standard negative feedback configuration is assumed. [30%]

(c) Consider the uncertain second order system

$$G_1 = \frac{1}{s^2+(2+\delta_1)s+0.2+\delta_2},$$

where δ_1 and δ_2 represent uncertainty in the damping and stiffness, respectively, and are only known to satisfy

$$|\delta_1| < 1, \; |\delta_2| < 1.$$

(i) Using the result in part (b), show that the constant controller $K = 0.8$ is guaranteed to stabilise G_1. [30%]

(ii) Find directly the precise range of values of δ_1 and δ_2 for which K stabilises G and explain why the small gain theorem is conservative. Describe how the small gain theorem may be extended to provide a less conservative result for this example. [20%]

END OF PAPER