ENGINEERING TRIPOS PART IIB

Thursday 29 April 2010 9 to 10.30

Module 4F3

NONLINEAR AND PREDICTIVE CONTROL

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

jmm03

1 (a) Define the following for an autonomous system

$$\dot{x} = f(x)$$

(i) Invariant set. Give two examples of invariant sets. [10%]

(ii) Stable equilibrium point. [5%]

(iii) Asymptotically stable equilibrium point. [5%]

(iv) Globally asymptotically stable equilibrium point. [5%]

(b) State Lasalle's Theorem and explain in what ways this can be more useful than Lyapunov's Theorem for proving asymptotic stability. [15%]

(c) Consider the system

$$\dot{z} = -\sum_{i=1}^{m} f_i(y_i)$$

$$\dot{y}_i = -h(z,y)y_i + z \qquad \text{for } i = 1,\ldots,m$$

where $y = [\ y_1 \ \ y_2 \ \ \ldots \ \ y_m\ ]$, $h(z,y) = 0$ at the origin $z = y_1 = \cdots = y_m = 0$ and $h(z,y) > 0$ otherwise. For $i = 1,\ldots,m$, functions f_i satisfy $f_i(0) = 0$, $y_i f_i(y_i) > 0$ for $y_i \neq 0$, and also f_i, h are Lipschitz continuous functions with f_i non-decreasing.

(i) By considering the function

$$V(z,y) = \alpha z^2 + \sum_{i=1}^{m} \int_0^{y_i} f_i(x)dx$$

where α is a constant to be chosen, show that the origin is an asymptotically stable equilibrium of the system. [40%]

(ii) If $f_i(y_i) = y_i^3$ for $i = 1,\ldots,m$, discuss whether a linearization of the system could be used to deduce asymptotic stability of the origin. [20%]

2 (a) Show that the describing function of the nonlinearity

$$f(e) = \begin{cases} 0 & \text{if } |e| \leq \delta \\ e-\delta & \text{if } e > \delta \\ e+\delta & \text{if } e < -\delta \end{cases}$$

with $\delta > 0$, is given by [35%]

$$N_\delta(E) = \begin{cases} 0, & \text{if } E \leq \delta \\ 1 - \frac{2}{\pi}\left[\sin^{-1}\left(\frac{\delta}{E}\right) + \frac{\delta}{E}\sqrt{1-\left(\frac{\delta}{E}\right)^2}\right] & \text{if } E > \delta \end{cases}$$

(b) Using your answer to part (a) find the describing function for the nonlinearity [15%]

$$g(e) = \begin{cases} 0 & \text{if } |e| \leq 1 \\ e-1 & \text{if } \ 1 < e \leq 2 \\ e+1 & \text{if } -2 \leq e < -1 \\ 1 & \text{if } e > 2 \\ -1 & \text{if } e < -2 \end{cases}$$

(c) The nonlinearity $f(e)$ is connected in negative feedback with a linear system whose transfer function is

$$G(s) = k\frac{s}{(s+2)^2}$$

(i) Find the values of k for which stability is guaranteed by the circle criterion. [25%]

(ii) Let k be within the range of values determined in (i). Are limit cycles predicted using the describing function method? Justify your answer. [15%]

(iii) Let k be within the range of values determined in (i). Will the interconnection still be stable if the nonlinearity $f(e)$ is replaced by $g(e)$? Justify your answer. [10%]

3 Consider the following linear discrete-time system

$$x(k+1) = Ax(k) + Bu(k)$$

where $x(k)$ is the state and $u(k)$ is the input. Let x_i and u_i be the predicted state and input, respectively, at time $k+i$, i.e. $x_0 = x(k)$ and $x_{i+1} = Ax_i + Bu_i$ for $i = 0, 1, \ldots$. Define

$$X := \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \quad U := \begin{bmatrix} u_0 \\ u_1 \end{bmatrix}.$$

(a) Find matrices Φ and Γ such that [20%]

$$X = \Phi x(k) + \Gamma U.$$

(b) Suppose that the following constraints are given:

$$Cu_i \le e \text{ for } i = 0, 1; \qquad Dx_1 \le f.$$

Compute matrices F and G and a vector g such that these constraints can be written as [20%]

$$FX + GU \le g$$

(c) Using the results of (a) and (b), find matrices S and T and a vector h, in terms of Φ, Γ, F, G and g, such that the constraints in (b) can be rewritten as

$$SU \le h + Tx(k)$$

and explain how this helps in the formulation of model predictive control (MPC) as a convex optimisation problem. [20%]

(d) Why is it important to formulate MPC as a convex optimisation problem, if possible? [20%]

(e) Cost functions in MPC are usually chosen to be quadratic. However, absolute-value costs are sometimes used, particularly to penalise inputs, for example:

$$J(x(k), U) = \sum_{i=1}^{2} \left(x_i^T Q x_i + |u_i| \right)$$

Give an example of an application in which such a cost may be more appropriate than a purely quadratic cost. [20%]

4 (a) Explain what is meant by *setpoint tracking*. Explain, with the aid of a block-diagram, how a target calculator can be used together with a model predictive regulator to obtain setpoint tracking. [20%]

(b) A system to be controlled is modelled as

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k) \\ y(k) &= Cx(k) \\ z(k) &= Hx(k) \end{aligned}$$

where $y(k)$ denotes a vector of measured outputs and $z(k)$ denotes a vector of controlled outputs.

(i) What is meant by an *offset-free target-equilibrium pair* for this system? [15%]

(ii) Explain why such a pair always exists if the matrix

$$\begin{bmatrix} I-A & -B \\ H & 0 \end{bmatrix}$$

has full row-rank. [15%]

(iii) What relationship between the number of inputs and the number of controlled outputs is necessary for this row-rank condition to hold? [10%]

(c) Now suppose that a piecewise-constant disturbance acts on the state:

$$\begin{aligned} x(k+1) &= Ax(k)+Bu(k)+B_d d(k) \\ d(k+1) &= d(k) \end{aligned}$$

Explain how the target calculator is modified in this case. [20%]

(d) Under what circumstances is it appropriate to implement the target calculator as an optimisation problem, such as a Quadratic Program? [20%]

END OF PAPER