ENGINEERING TRIPOS PART IIB

Monday 19 April 2010 9 to 10.30

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

sss04

1 (a) Echo cancellation is an important application for telephony. A hands-free unit includes a microphone and a loudspeaker. The voice of the remote speaker, coming out of the loudspeaker, is reflected by the room (the echo), and sent back through the microphone to the remote speaker. The echo is annoying to the remote speaker and must be cancelled.

Describe how you would use the Recursive Least Squares algorithm to design an adaptive echo canceller for this application. Draw diagrams as appropriate and define all relevant signals and the cost function. [30%]

(b) An adaptive nonlinear filter has the following structure:

$$y(n) = \sum_{k=0}^{M-1} h_k(n)u(n-k) + \sum_{i=0}^{M_1-1} \sum_{j=i}^{M_2-1} h_{i,j}(n)u(n-i)u(n-j)$$

where $\{u(n)\}_{n=0}^{\infty}$ is the input sequence to the nonlinear filter which has coefficients $\{h_k(n)\}_{k=0}^{M-1}$ and $\{h_{i,j}(n)\}_{i=0,j=i}^{M_1-1,M_2-1}$. The coefficients evolve with time n as the filter is adaptive. Let $\{d(n)\}_{n=0}^{\infty}$ be the desired response, $M = 2$, $M_1 = 1$ and $M_2 = 2$. Derive the LMS algorithm to update the filter coefficients so that $E\left\{(d(n) - y(n))^2\right\}$ is minimised. What condition must be placed on the step-size μ so that the filter converges in mean? State clearly any assumptions you make. [35%]

(c) Describe what happens if the third order statistics of $\{u(n)\}$ are zero, i.e.

$$E\{u(n)^3\} = E\{u(n)^2u(n-1)\} = E\{u(n)u(n-1)^2\} = 0.$$

Describe in detail an LMS implementation that uses two step-sizes, μ_1 and μ_2, and state the conditions so that the filter converges in mean. Discuss how this new LMS algorithm might improve performance over the single step-size LMS algorithm. [35%]

2 We have repeated observations of a random variable x through

$$y(n) = x + v(n) \quad \text{for } n = 1, 2, \ldots$$

where $\{v(n)\}$ is an independent and identically distributed zero-mean scalar noise sequence, independent of x, with variance $E\left\{v(n)^2\right\} = \sigma_v^2$. Also, $E(x) = 0$ and $E\left(x^2\right) = \sigma_0^2$.

(a) Let $\widehat{x}(n-1)$ be an estimate of x using $\{y(1), \ldots, y(n-1)\}$. Upon receiving $y(n)$, the estimate of x is to be updated as follows

$$\widehat{x}(n) = K(n)\left(y(n) - \widehat{x}(n-1)\right) + \widehat{x}(n-1).$$

Find the value of the gain $K(n)$ that minimises the mean square error $E\left((\widehat{x}(n) - x)^2\right)$. Carefully detail your derivation and assumptions made. [50%]

(b) The Kalman filter for estimating x can be expressed as

$$\widehat{x}(n) = \widehat{x}(n-1) + \frac{\sigma_0^2}{n\sigma_0^2 + \sigma_v^2}\left(y(n) - \widehat{x}(n-1)\right).$$

Using this equation find an expression for $\widehat{x}(n)$ in terms of $\widehat{x}(0)$ and $y(1), \ldots, y(n)$. [30%]

(c) Compute the variance of the Kalman filter estimate and that of the sample mean estimate,

$$\frac{1}{n}\sum_{i=1}^{n} y(i),$$

and conclude which estimator is better. [20%]

3 The estimate of the power spectrum of a random process $\{x_n\}$ is being recursively updated as follows:

$$\widehat{S}^{(k)}(e^{j\omega}) = (1-\gamma_k)\widehat{S}^{(k-1)}(e^{j\omega}) + \gamma_k \frac{1}{N}\left|\sum_{n=0}^{N-1} x_n^{(k)} e^{-jn\omega}\right|^2$$

where $x_n^{(k)} = x_{n+N(k-1)}$ is the kth sequence of N data points. This equation is initialised at $k=1$ with $\widehat{S}^{(0)} = 0$.

(a) Discuss this approach to power spectrum estimation. In your discussion, address the choice of N, as well as the following two choices for the sequence $\{\gamma_k\}$,

$$\gamma_k = k^{-1} \quad \text{and} \quad \gamma_k = \alpha,$$

where $0 < \alpha < 1$. [15%]

(b) Assume that successive periodograms are uncorrelated. Obtain the variance of $\widehat{S}^{(k)}$ when $\{x_n\}$ is a zero mean Gaussian process for the choices of γ_k in part (a). Now evaluate the limiting values of these variance expressions. [45%]

(c) Let

$$x_n = A\sin(\omega_1 n + \phi_1) + B\sin(\omega_2 n + \phi_2) + v_n$$

where A, B, ω_1, ω_2 are constants and ϕ_1, ϕ_2 are independent random phases uniformly distributed between 0 and 2π, and $\{v_n\}$ is a zero mean white noise sequence which is independent of ϕ_1, ϕ_2. What is the minimum length N to be used for each sequence to resolve the two frequencies reliably if $\omega_2 - \omega_1 \geq 0.05\pi$? [You may assume that two frequencies can be resolved if in the expected value of the periodogram, the 6dB central lobe bandwidth of one component does not overlap with that of the other component. Also, the 6dB central lobe bandwidth is measured from the middle of the central lobe to the 1/4 power point. For a rectangular window, this value is $1.78(2\pi/M)$ where M is the window length.] [35%]

(d) Why is it not advantageous to increase N beyond the value found in (c)? [5%]

4 (a) Discuss the periodogram method and the maximum likelihood method for power spectrum and signal model estimation. [20%]

(b) The ARMA(P,Q) model is

$$x_n = -\sum_{p=1}^{P} a_p x_{n-p} + \sum_{q=0}^{Q} b_q w_{n-q}$$

where $\{w_n\}$ is a sequence of independent and identically distributed random variables with mean zero and unit variance. The estimated autocorrelation function of $\{x_n\}$ at lags $k = 0,1,2,3,4$ are

$$\widehat{R}_{XX}[0] = 2, \quad \widehat{R}_{XX}[1] = 1, \quad \widehat{R}_{XX}[2] = -1, \quad \widehat{R}_{XX}[3] = 0.5, \quad \widehat{R}_{XX}[4] = 0. \quad (1)$$

Estimate $\{a_p\}_{p=1}^{P}$ and $\{b_q\}_{q=0}^{Q}$ using the values in (1) for each of the following cases and state clearly any assumptions you make. (See hint below.)

(i) $P = 0$ and $Q = 1$ using the spectral factorization method. [35%]

(ii) $P = 1$ and $Q = 1$. [40%]

(c) State which of the two models in (b) is more appropriate for the data and justify your answer. [5%]

[Hint: You may use the following information: $R_{XX}[r] = E[x_n x_{n+r}]$ is

$$R_{XX}[r] = -\sum_{p=1}^{P} a_p R_{XX}[r-p] + c_r$$

where

$$c_r = \begin{cases} \sum_{q=r}^{Q} b_q h_{q-r} & \text{if } r \le Q \\ 0 & \text{if } r > Q \end{cases}$$

and $\{h_n\}$ is the impulse response of the linear filter generating $\{x_n\}$.]

END OF PAPER

4F7 Digital filters and spectrum estimation
List of numerical answers to 2010 exam

Question 1
Part (a): –
Part (b): –

Part (c): –
Question 2
Part (a):

$$K(n) = \frac{\sigma(n-1)^2}{\sigma_v^2 + \sigma(n-1)^2}.$$

Part (b):

$$\widehat{x}(n) = \frac{\sigma_0^2}{n\sigma_0^2 + \sigma_v^2} \sum_{i=1}^{n} y(i).$$

Part (c) Sample mean:

$$E\left\{\left(\frac{1}{n}\sum_{i=1}^{n} y(i)\right)^2\right\} = \frac{\sigma_v^2}{n} + \sigma_0^2.$$

Kalman estimate:

$$E\left\{\widehat{x}(n)^2\right\} = \left(\frac{\sigma_0^2}{n\sigma_0^2 + \sigma_v^2}\right)^2 \left(n\sigma_v^2 + n^2\sigma_0^2\right).$$

Since $\frac{\sigma_0^2}{n\sigma_0^2+\sigma_v^2} < \frac{1}{n}$, Kalman is better.

Question 3
Part (a): –
Part (b): –
Part (c): $N \geq 143.4/2 = 71.7$.
Part (d): –

Question 4
Part (a): –
Part (b-i): $b_0 = 1$, $b_1 = 1$.
Part (b-ii): $a_1 = 1$, $b_0 = \sqrt{3}$, $b_1 = -\sqrt{3}$.
Part(c): –