ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Monday 26th April 2010 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Data Sheet for 4M12 (3 sides).

STATIONERY
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

tph05

1 (a) Express the biharmonic operator $\nabla^4 w = \nabla . \nabla (\nabla . \nabla w)$ using index notation. Comment on any symmetries. [15%]

(b) Show that $\varepsilon_{ipq}\,\varepsilon_{lpq} = 2\,\delta_{il}$. [30%]

(c) Show that, for any vector $\boldsymbol{w}$, there exists a unique skew-symmetric matrix $\boldsymbol{W}$ such that

$$\boldsymbol{w} \times \boldsymbol{v} = \boldsymbol{W}\,\boldsymbol{v}$$

for all vectors $\boldsymbol{v}$.

Note that if $\boldsymbol{W}$ is skew-symmetric, then $\boldsymbol{W}^T = -\boldsymbol{W}$. [35%]

(d) For the functional

$$J = \int_V \left(\nabla^2 u\right)\left(\nabla^2 u\right) dV ,$$

use the directional derivative method to prove that a stationary point of the functional is also a minimum point for non-trivial u. [20%]

2 (a) If the function u minimises

$$J = \frac{1}{2}\int_V \nabla u.\nabla u \, dV - \int_V f u \, dV \,, \qquad (1)$$

(i) Show that u satisfies

$$-\nabla^2 u = f$$ [10%]

(ii) Suggest boundary conditions which are consistent with equation (1) and would lead to a stable result. [20%]

(b) In a particular optimal control problem, for a given w, we wish to find f such that the 'cost' functional

$$I = \frac{1}{2}\int_V (u-w)^2 + f^2 \, dV$$

is minimised subject to the constraint on u that it satisfies $-\nabla^2 u = f$ with $u = 0$ on the boundary ∂V.

Using a Lagrange multiplier denoted by λ,

(i) Write down the functional J whose stationary points correspond to minimising I subject to the constraint on u. [25%]

(ii) Show that finding the stationary points of J is equivalent to solving

$$-\nabla^2 u = f$$

and

$$-\nabla^2 \lambda = u - w$$

with $\lambda = 0$ on ∂V and $f = -\lambda$. [45%]

3 (a) Explain how equations of the form

$$a(x,y,u)\frac{\partial u}{\partial x} + b(x,y,u)\frac{\partial u}{\partial y} = c(x,y,u)$$

can be solved by integrating along characteristic lines. [20%]

(b) The function u satisfies the equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 2 \quad \text{for } t > 0 \text{ and all } x, \qquad (1)$$

with

$$u(x,0) = \begin{cases} 0 & \text{for } x < 0 \\ x & \text{for } 0 \le x \le 1 \\ 1 & \text{for } x > 1 \end{cases}$$

By integrating along characteristic lines, or otherwise, find $u(x,t)$. You should explain carefully to which parts of the region $t > 0$, any different expressions for u that you obtain apply. [50%]

(c) Explain carefully how discontinuities can form for solutions to equations like equation (1) from part (b). Do discontinuities form in this case ? [30%]

4 (a) Explain what is meant by a well-posed problem in the context of second order partial differential equations. [15%]

(b) Explain how the Maximum Principle ensures that the problem

$$\nabla^2 u = 0$$

in a region V, with $u = U(\underline{x})$ for $\underline{x}$ on the boundary S surrounding V will be well-posed. [15%]

(c) The function ϕ satisfies

$$\nabla^2 \phi = f$$

in a region V, with $\dfrac{\partial \phi}{\partial n} = \psi(\underline{x})$ on the boundary S surrounding V. Describe any extra conditions that must be applied to make this problem well-posed. [20%]

(d) The vector $\underline{r}$ satisfies the system of equations

$$\frac{\partial}{\partial t}\underline{r} + A\frac{\partial}{\partial x}\underline{r} = 0$$

for $t \geq 0$ and $0 \leq x \leq L$, where the matrix A is given by:

$$A = \begin{bmatrix} u & c & 0 \\ c & u & 0 \\ 0 & 0 & u \end{bmatrix}$$

and where $u = u(x,t)$ and $c = c(x,t)$.

(i) Show that this system of equations is hyperbolic and determine the characteristic speeds. [20%]

(ii) If all boundary conditions for this problem are Dirichlet ones (i.e. specified values), describe how these boundary conditions must be formulated to make this problem well-posed for the case where both u and c are positive and constant. [30%]

End of Paper

Answers

1. (a) $\dfrac{\partial^4 w}{\partial x_i \partial x_i \partial x_j \partial x_j}$ (c) $W_{ik} = \varepsilon_{ijk} w_j$

2. (b) (i) $J = \dfrac{1}{2}\int_V (u-w)^2 + f^2 \, dV + \int_V \lambda\left(\nabla^2 u + f\right) dV$

3. $$u(x,t) = \begin{cases} 2t & x < t^2 \\ 2t + \dfrac{x-t^2}{t+1} \quad \text{for} & t^2 < x < t^2 + t + 1 \\ 1 + 2t & t^2 + t + 1 < x \end{cases}$$

4. (d) (i) u, $u - c$ and $u + c$