ENGINEERING TRIPOS PART IIB
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Tuesday 4 May 2010 2:30 to 4

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

Answer not more than **three** *questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:

4M13 Datasheet (4 pages).

Answers to Sections A and B should be tied together and handed in separately.

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
Graph paper	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

GC05

SECTION A

1 Figure 1 shows the streamlines of an ideal (inviscid and irrotational) fluid flowing into a 90° corner formed by the positive x and y axes, which are drawn using thick black lines.

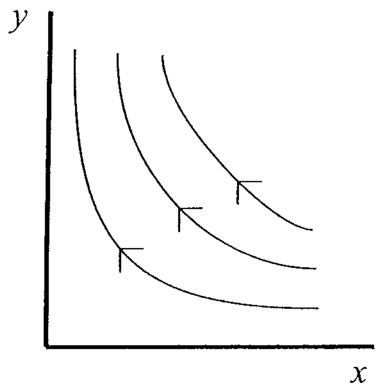


Fig. 1

(a) Identify a simple conformal mapping $G(z)$ that maps the thick boundary onto the real axis of the complex plane. [30%]

(b) Hence determine the velocity field $\mathbf{v}(x,y) \equiv \left(v_x(x,y), v_y(x,y)\right)$ of this fluid flow near the corner. [50%]

(c) By considering the velocity field found in (b) in the limit $x \longrightarrow \infty$, comment on the validity of the result far away from the corner. [20%]

2 Using contour integration, evaluate the following integrals;

(a) $\int_{-\infty}^{\infty} \cos(\lambda x)/(x^2+1)\, dx, \quad \lambda > 0$ [50%]

(b) $\int_0^{\infty} \sin(x)/x\, dx$ [50%]

SECTION B

3 (a) Explain how *slack variables* can be used to convert inequality constraints into equality constraints in linear programming problems. [10%]

(b) A mining company operates three types of mine: opencast, mechanized and labour-intensive. The operating details to produce one million tonnes of coal per year at each type of mine are given in the table below.

	Opencast	Mechanized	Labour-intensive
Workers (thousands)	2	5	10
Cutting machines	4	2	1
Annual profit (£M)	10	5	–1

The mining company has 70,000 workers and 50 cutting machines available. For political reasons all 70,000 workers must be used and at least five million tonnes of coal each year must be produced at labour-intensive mines. It is not necessary to use all the cutting machines.

(i) Formulate the task of finding the number of mines of each type that maximizes annual profit as a linear programming problem in standard form. [20%]

(ii) Identify a suitable feasible initial solution to this problem. [10%]

(iii) Solve this problem using the simplex method. (You may treat integer variables as continuous ones for this purpose.) Hence show that the maximum profit the mining company can make is £95M per year. [50%]

(iv) How is the optimum affected by the number of cutting machines available? [10%]

4 A biofuels company is considering setting up production facilities in each of n different regions. The company is committed to providing h_i jobs in region i, $i = 1, 2, \ldots, n$. The directors wish to allocate their initial capital investment k in such a way as to maximise total annual biofuel production. Each factory's annual production p_i follows the Cobb-Douglas production function, $p_i = a_i\sqrt{x_i h_i}$, where x_i is the capital investment in factory i and a_i is a constant, the value of which depends on the region.

(a) Formulate the task of allocating the initial capital investment optimally between the n factories as a constrained optimization problem in standard form. [10%]

(b) Use the Lagrange multiplier method to find the optimal allocation of capital investment in terms of the values of a_i, h_i and k. [40%]

(c) The company's finance director suggests investing an amount b of the capital in bonds yielding an annual interest rate α rather than in the factories. Suppose that a profit p is made on every unit of biofuel produced. If the objective is now to maximize profit, show that a suitable objective function to be minimized is

$$f = -\alpha b - p\sum_{i=1}^{n} a_i\sqrt{x_i h_i}$$

and that the optimal investment in each factory $x_i = \dfrac{p^2 a_i^2 h_i}{4\alpha^2}$. How much capital should be invested in bonds? [50%]

END OF PAPER

4M13

OPTIMIZATION

DATA SHEET

1. Taylor Series Expansion

For one variable:

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

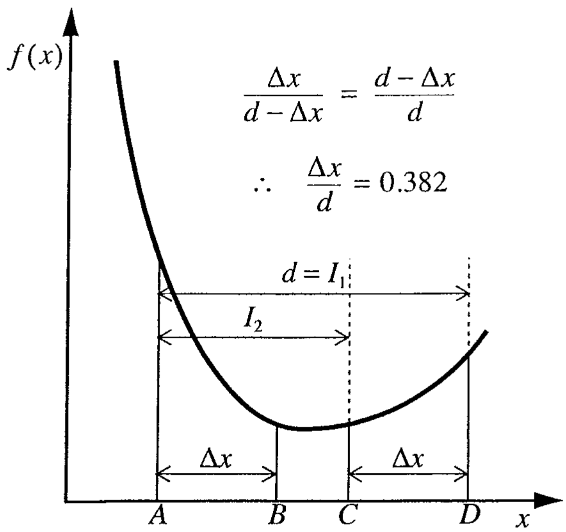
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T(\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T H(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } H(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$H(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. Golden Section Method



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1}\nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T\mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k)\mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T\mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\,\mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k\mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|}\right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k\mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T\nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\,\mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T\mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\,J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1} J(\mathbf{x}_k)^T\mathbf{r}(\mathbf{x}_k)$

$$\text{where } \boldsymbol{J}(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} = 0 \quad (n \text{ equations})$$
$$\mathbf{h}(\mathbf{x}^*) = 0 \quad (m \text{ equations})$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \left[\nabla h_1(\mathbf{x}^*) \ \ldots \ \nabla h_m(\mathbf{x}^*) \right] = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \le 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} = 0 \quad (n \text{ equations})$$
$$\mathbf{h}(\mathbf{x}^*) = 0 \quad (m \text{ equations})$$
$$\forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) = 0 \quad (p \text{ equations})$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \ge 0$ are the Kuhn-Tucker multipliers.

9. **Penalty & Barrier Functions**

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \le 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0, g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

$$\text{minimise } q(\mathbf{x}, p_k)$$