1 (5)

a) Hydro static balance : $\frac{\partial p}{\partial z} = -\rho g$

I & IInd Law of Thermo-dynamics

$$dQ = Tds = dh - vdp$$
$$= dh - \frac{1}{\rho} dp$$

for adiabatic case $dQ = 0$

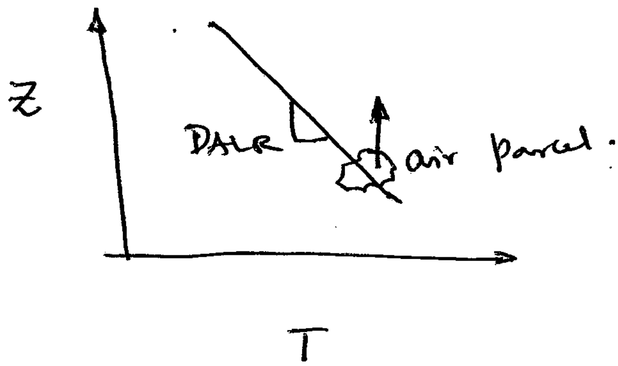
$$\Rightarrow dh = \frac{1}{\rho} dp = c_p dT$$

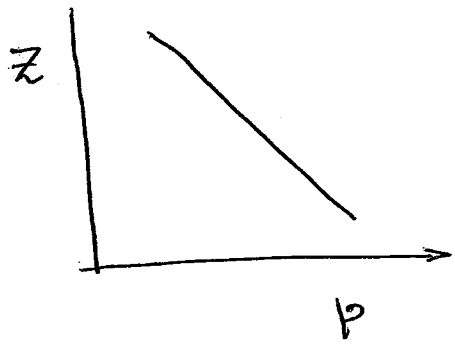
$$\Rightarrow \boxed{\frac{\partial T}{\partial z} = -\frac{g}{c_p}}$$ using hydro static balance.

as required.

Significance:

This gives the rate @ which the temperature drops with height in the atmosphere. This is given as in the fig. below.





This condition Corresponds to the neutral stability of the atmosphere, as follows.

Let us say the air parcel is moved up as shown above adiabatically, then

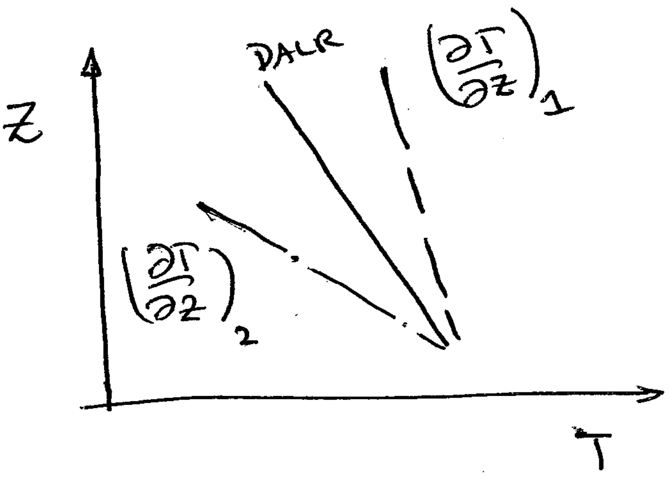
$p \downarrow$ (See the fig. on the right, above)

$\Rightarrow$ $\rho \downarrow$ $\Rightarrow$ air parcel will move up further by bouyancy.

But $T \downarrow \Rightarrow \rho \uparrow \Rightarrow$ the air parcel will try to move back.

$\Rightarrow$ For, DALR the perturbed air parcel will stay in its new position. $\Rightarrow$ Neutrally Stable condition

(NOTE: this is optional, in case a student starts discussing the qualitative implications of DALR)



in the figure above two cases are shown

$$\left(\frac{\partial T}{\partial z}\right)_1 \quad - \quad \text{Stable}$$

$$\left(\frac{\partial T}{\partial z}\right)_2 \quad - \quad \text{unstable.},$$

(explanatory)

(b) (i) $$\theta = T\left(\frac{p_0}{p}\right)^{k} \qquad k = \frac{R}{C_p}$$

$$d\theta = P' dT + T dP'$$

where $P' = \left(\frac{p_0}{p}\right)^{k}$

$$\Rightarrow \quad dP' = -k\left(\frac{p_0}{p}\right)^{k-1} \frac{p_0}{p^2}\, dp$$

$$= -k P' \frac{1}{p}\, dp.$$

$$d\theta = P' dT - \frac{RT}{c_p} P' \frac{1}{p} dp$$

$$\Rightarrow \frac{\partial \theta}{\partial z} = P' \frac{\partial T}{\partial z} - \frac{RT}{c_p} P' \frac{1}{p} \frac{\partial p}{\partial z}$$

$$= P' \left(\frac{\partial T}{\partial z} + \frac{g}{c_p} \right)$$

$$\Rightarrow \boxed{\frac{\partial \theta}{\partial z} = P' \left(\Gamma_{DALR} - \Gamma_{act} \right)}$$

where $\Gamma_{act} = -\left(\frac{\partial T}{\partial z} \right)_{act}$.

$\Rightarrow$ when $\Gamma_{act} < \Gamma_{DALR}$ $\quad \frac{\partial \theta}{\partial z} > 0$ stable.

$\Gamma_{act} = \Gamma_{DALR}$ $\quad \frac{\partial \theta}{\partial z} = 0$ Neutral

$\Gamma_{act} > \Gamma_{DALR}$ $\quad \frac{\partial \theta}{\partial z} < 0$ unstable.

This is for a calm day because no wind velocity is involved.

(ii) $$T = A + B e^{-z/H}$$

$$\frac{\partial T}{\partial z} = -\frac{B}{H} e^{-z/H} = \left(\frac{\partial T}{\partial z}\right)_{act}.$$

$$\Rightarrow \quad \Gamma_{act} = \frac{B}{H} e^{-z/H}$$

for stability $\frac{\partial \theta}{\partial z} > 0$ from (i)

$$\Rightarrow \quad \Gamma_{act} < \Gamma_{DALR}$$

$$\frac{B}{H} e^{-z/H} < \frac{g}{c_p}$$

when $z = H$ $\boxed{B < \frac{Hge}{c_p}}$

for large z $\boxed{B < \frac{g}{c_p} H e^{z/H}}$

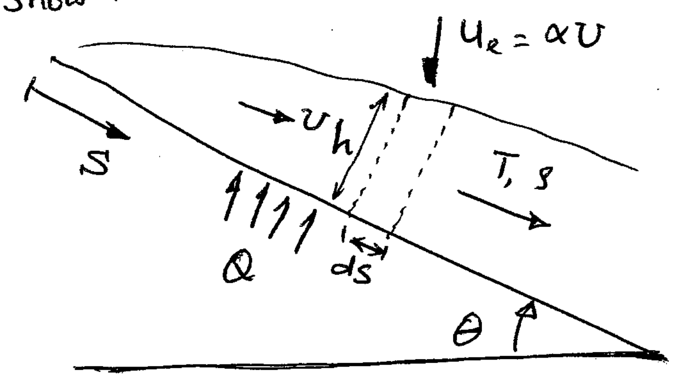
$\therefore$ for $z > H$.
although $e^{-z/H}$ is small
$H e^{z/H}$ can be of order one.

Examiner's comment:

Part (a), which was based on the lecture notes, were very well answered, while Part (b) that demanded some analysis was poorly done.

2

(a)



p_a, ρ_a, T_a

$\downarrow g$

unit width in the perpendicular direction.

$\Delta\rho = (\rho_a - \rho)$

entrainment is in the normal direction.

(b) Control volume as shown above.
Take the situation to be steady.

mass:

Rate of change = in − out

$$0 = \rho U h + \rho_a \alpha U ds - \rho U h - \frac{d}{ds}(\rho U h) ds$$

$$\Rightarrow \boxed{\frac{d}{ds}(\rho U h) = \rho_a \alpha U} \quad — \quad (1)$$

momentum: (along s)

$$0 = \rho U^2 h - \rho U^2 h - \frac{d}{ds}(\rho U^2 h) ds - \underbrace{\Delta\rho\, h\, ds\, g \sin\theta}_{\text{Bouyancy force.}}$$

$$\Rightarrow \frac{d}{ds}(\rho U^2 h) = -\Delta\rho\, g h \sin\theta.$$

$$\boxed{\frac{d}{ds}(\rho U^2 h) = (\rho - \rho_a) g h \sin\theta} \quad — \quad (2)$$

energy:

$$0 = h\rho U c_p \Delta T - \rho u c_p h \Delta T - \frac{d}{ds}(\rho u c_p h \Delta T) + Q\,ds.$$

$$\Rightarrow \boxed{\frac{d}{ds}\left[\rho U c_p h (T - T_a)\right] = Q.}$$

Assume $\frac{\Delta\rho}{\rho_a} = \frac{\rho_a - \rho_\infty}{\rho_a} << 1 \Rightarrow \boxed{\rho \simeq \rho_a}$ for convective part. (Boussinesq approximation)

$$\Rightarrow \boxed{\frac{d}{ds}\left[Uh(T - T_a)\right] = \left(\frac{Q}{\rho_a c_p}\right) = \text{const.}} \quad - (3)$$

(c) $h = As^a$, $U = Bs^b$, $(T - T_a) = Cs^c$

(i) $(\rho - \rho_a) = Ds^d$

The pressure must be continuous across the entrainment interface.

$\Rightarrow p \simeq p_a$

using ideal gas law

$$(T - T_a) = \frac{P}{R}\left(\frac{1}{\rho} - \frac{1}{\rho_a}\right) = -\frac{p_a}{R}\frac{(\rho - \rho_a)}{\rho_a^2}$$

$$\Rightarrow \boxed{d = c}$$

from ①

$$\frac{d}{ds}\left(ABS^{a+b}\right) = \alpha A S^{b}.$$

$\Rightarrow \quad a+b-1 = b$ by equating powers of S

$\Rightarrow \quad \boxed{a = 1}$

from ②

$$2b + a - 1 = c + a$$

$\Rightarrow \quad \boxed{2b - c = 1}$

from ③ $\quad b + c = 0$

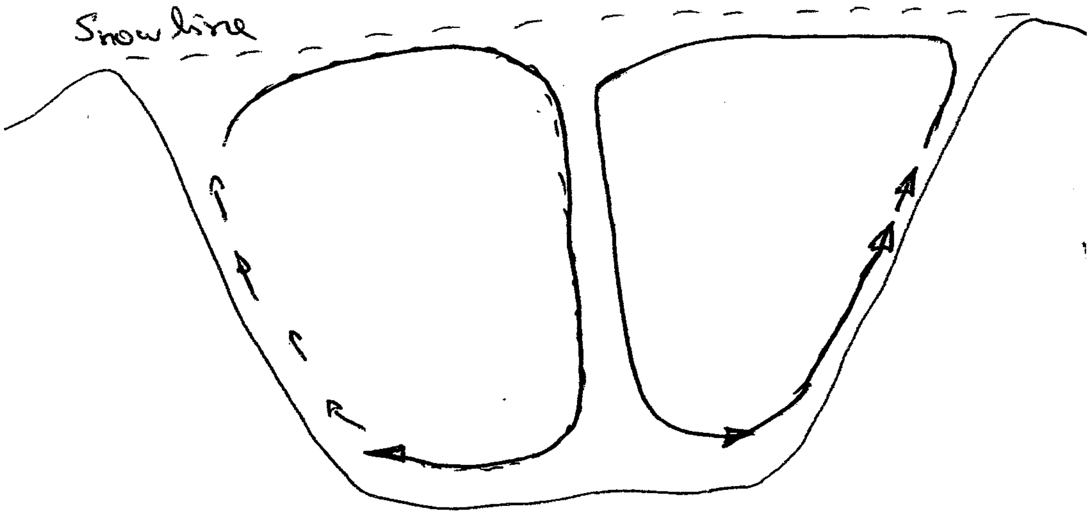
Solving the above two equations

$$\boxed{\begin{aligned} b &= 1/3 \\ c &= -1/3 = d. \end{aligned}}$$

$\Rightarrow \quad U = B S^{1/3}$

$$T - T_a = c\, S^{-1/3}$$

(ii) @ day time the slope is heated by sunshine.



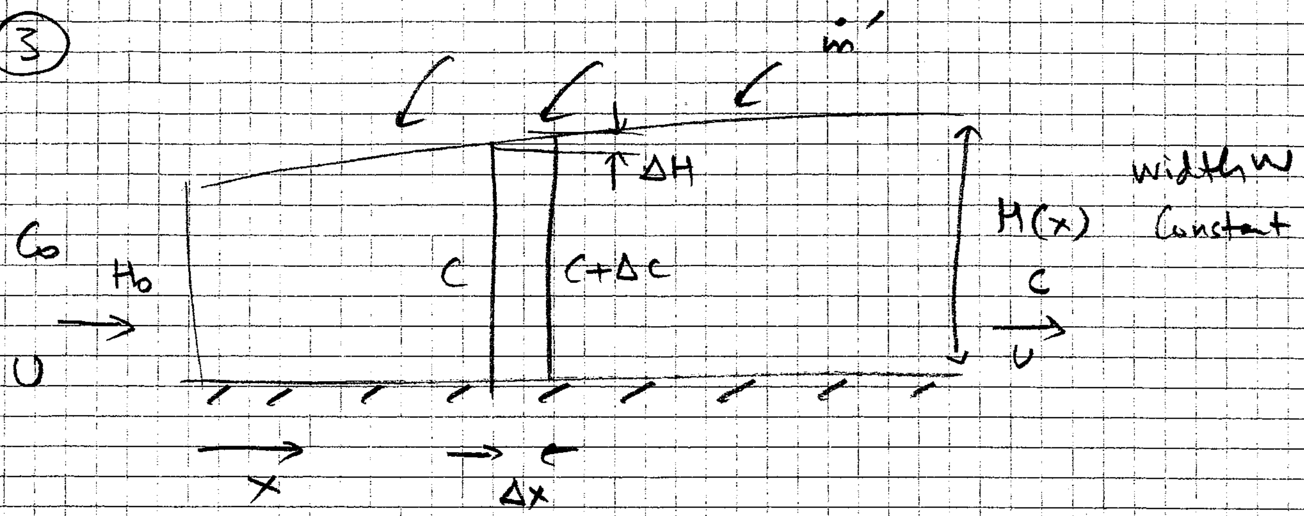
@ night time, the slopes are cooled the flow will reverse.

near the snow line, the ground will be cooler and thus the flow will be trapped as shown above.

Examiner's comment:

Very few students could derive the governing equations (especially the energy equation) rigorously. The power law expressions were input correctly, but the subsequent analysis was virtually always unsuccessful.

③



- Conservation of mass (at steady state)

$$(\text{Mass in}) - (\text{Mass out}) = 0$$

$$\rho UWH - \rho UW(H+\Delta H) + \dot{m}'\Delta x = 0$$

$$\Rightarrow \frac{\dot{m}'}{\rho}\Delta x = UW\Delta H$$

$$\Leftrightarrow \frac{dH}{dx} = \frac{\dot{Q}}{UW} \qquad \dot{Q} = \dot{m}'/\rho$$

$$\Rightarrow H = H_0 + (\dot{Q}/UW)\,x$$

- Conservation of pollutant:

$$\underbrace{UWHC}_{C\text{-in}} - \underbrace{UW(H+\Delta H)(C+\Delta C)}_{C\text{-out}} - \underbrace{k(WH\Delta x)C}_{\text{destruction of } C} = 0$$

$$\Leftrightarrow \cancel{UWHC} - \cancel{UWHC} - UWH\Delta C - UW\Delta HC - \underbrace{\cancel{UW\Delta H\Delta C}}_{\text{second-order term}} - kWH\Delta xC = 0$$

$$\Leftrightarrow -UW\Delta HC - UWH\Delta C - kWH\Delta x\,C = 0$$

$$\Leftrightarrow \frac{dC}{dx} = -\left(\frac{1}{H}\frac{dH}{dx} + \frac{k}{U}\right)C \qquad \text{QED}$$

(b) If $\dot{m}' = 0$, $\frac{dH}{dx} = 0$

$$\Rightarrow \frac{dc}{dx} = -\frac{k}{v} c$$

$$\Rightarrow C = C_o \exp\left(-\frac{k}{v}x\right)$$

$\Rightarrow$ c will be below the safe limit $\left(\frac{C_o}{10}\right)$

at $-\frac{kx}{v} = \ln(0.1) \Rightarrow x = \frac{2.3U}{k}$

If $k=0$, $\frac{dc}{dx} = -\frac{1}{H}\frac{dH}{dx} c$

$$\Rightarrow \frac{dc}{c} = -\frac{(\dot{Q}/UW)}{H_o + (\dot{Q}/UW)x} dx$$

Let $A = \dot{Q}/UWH_o$

$$\Rightarrow \frac{dc}{c} = -\frac{A}{1+Ax} dx$$

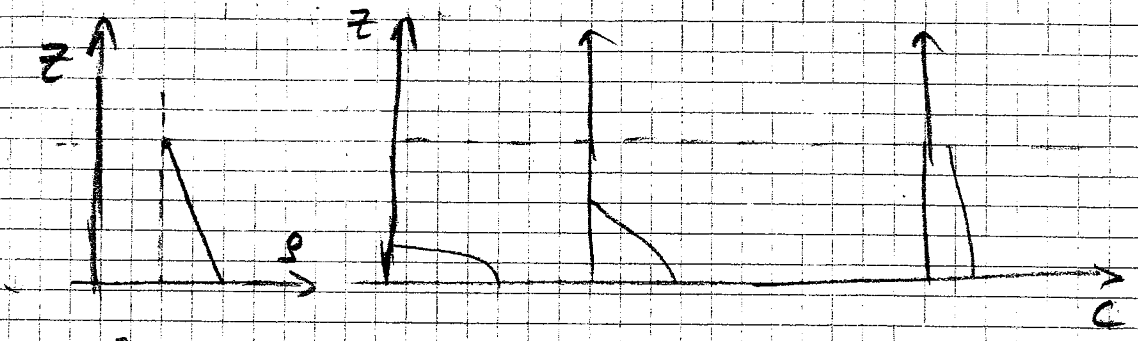
$$\Rightarrow \ln\frac{c}{C_o} = -\ln(1+Ax)$$

$$\Rightarrow \frac{C}{C_o} = \frac{1}{1+Ax}.$$

$\Rightarrow$ c will read safe limit when

$$1 + Ax = 10 \Rightarrow x = 9/A = \frac{9 \cdot UWH_o}{\dot{Q}}$$

(c)



(exaggerated)

density stratification

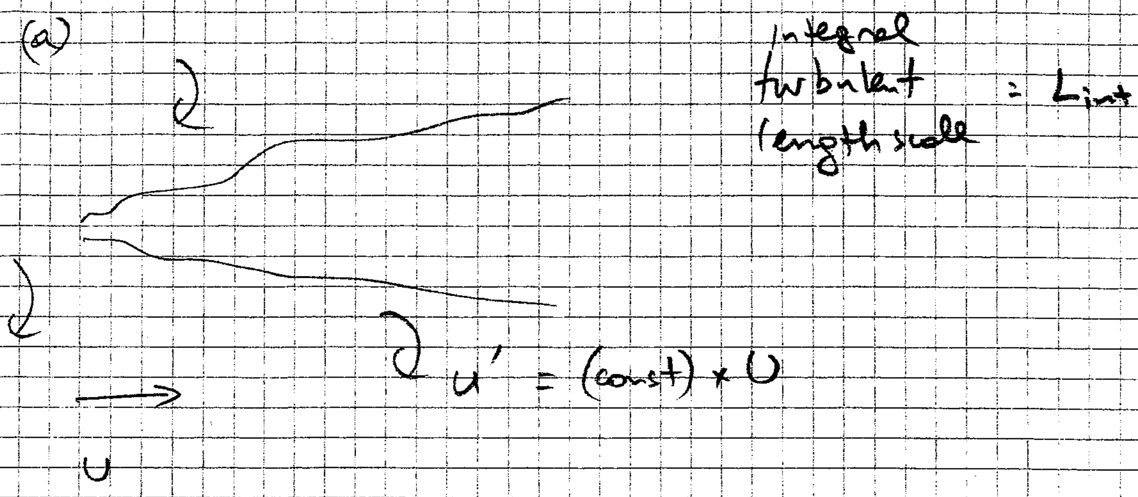
Desity stratification will reduce the turbulence created in the boundary layer at the bottom ⇒ pollutant will not spread too quickly and hence assumption of cross-stream uniformity may not be valid.

At different x, c distribution in z will be like a Gaussian plume. Eventually, c will be uniform.

Examiner's comment:

The most popular question, generally well done. The final discussion on effects of stratification on mixing was not as well answered as expected.

4 (a)



Initially, the plume is spreading by the action of eddies of size up to the plume's size, σ. Eddies that are bigger than σ just convect ("flap") the plume. Therefore:

Diffusivity $K = \underbrace{A \times (\text{const})}_{C} \times \overbrace{(\text{const}) \times U}^{u'} \times \sigma \leftarrow$ early

$= A \times (\text{const}) \times U \times L_{int} \leftarrow$ late

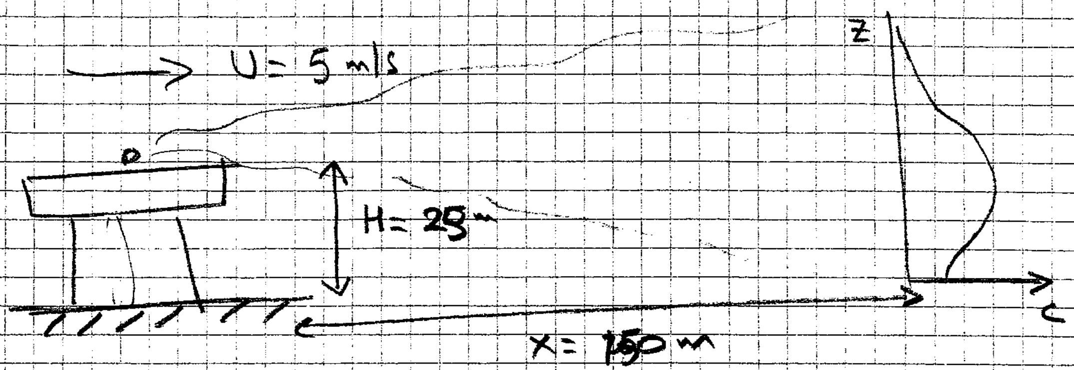
(A: some constant)

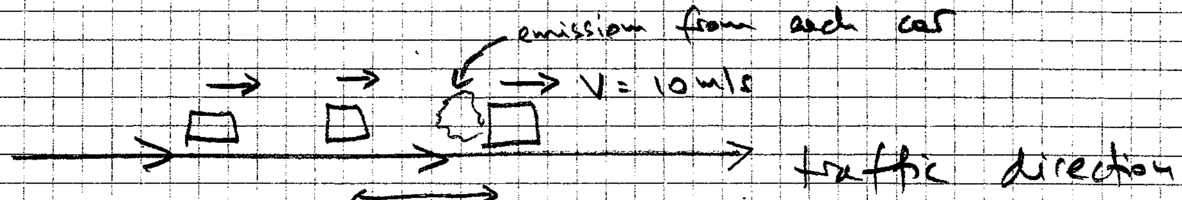
$\Rightarrow$ early mixing: $\sigma^2 = \dfrac{2x}{U} C U \sigma \Rightarrow \sigma \sim x$

late mixing: $\sigma^2 = \dfrac{2x}{U} C U L_{int} \Rightarrow \sigma \sim x^{1/2}$

Vertical dispersion coefficient is smaller because the vertical velocity fluctuations are smaller than the horizontal ones in the atmospheric boundary layer & also the lengthscale is constrained.

(b)





average distance between cars = 10 m $\Rightarrow$ # of cars per metre = 0.1 m^{-1}

Average emission from traffic = $(1 \text{ g/s}) \times \left(\frac{\text{cars per}}{\text{m}}\right)$

$$\Rightarrow (Q/L) = 0.1 \times 10^{-3} \ \frac{\text{kg}}{\text{m s}}$$

Emission from line source at height H with ground effect:

$$\bar{\Phi}(x,z) = \frac{(Q/L)}{U} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma_z} \left(\exp\left[-\frac{(z-H)^2}{2\sigma_z^2} \right] + \exp\left[-\frac{(z+H)^2}{2\sigma_z^2} \right] \right)$$

At $x = 150$ m, from chart; vertical dispersion coefficient is $\simeq 20$ m for neutral atmosphere (class D); using urban values

$$\Rightarrow \bar{\Phi}(z) = A \exp\left[-\frac{(z-H)^2}{2\sigma_z^2} \right] + A \exp\left[-\frac{(z+H)^2}{2\sigma_z^2} \right]$$

$$A \simeq 4 \times 10^{-7} \text{ kg/m}^3$$

To find z at which $\bar{\Phi}$ is maximized, set

$$\frac{d\bar{\Phi}}{dz} = 0 \Rightarrow -\frac{2(z-H)}{2\sigma_z^2} A \exp\left(-\frac{(z-H)^2}{2\sigma_z^2}\right) - \frac{2(z+H)}{2\sigma_z^2} A \exp\left(-\frac{(z+H)^2}{2\sigma_z^2}\right) = 0$$

$$\Leftrightarrow (H-z) \exp\left(-\frac{(z-H)^2}{2\sigma_z^2}\right) = (H+z) \exp\left(-\frac{(z+H)^2}{2\sigma_z^2}\right)$$

Using $H = 25$ m, $\sigma_z = 20$ m, $z = 22$ m, it can be shown that LHS = RHS.

At $z = 22$ m, $\bar{\Phi} \simeq 4.2 \times 10^{-7}$ kg/m^3

Examiners' comment:

The theory behind diffusivity was well understood and the application of the Gaussian plume equation was done well. However, the analytical part became a little messy.