ENGINEERING TRIPOS PART IIB

2011

Module 4A10

FLOW INSTABILITY

Solutions

STATIONERY REQUIREMENTS SPECIAL REQUIREMENTS

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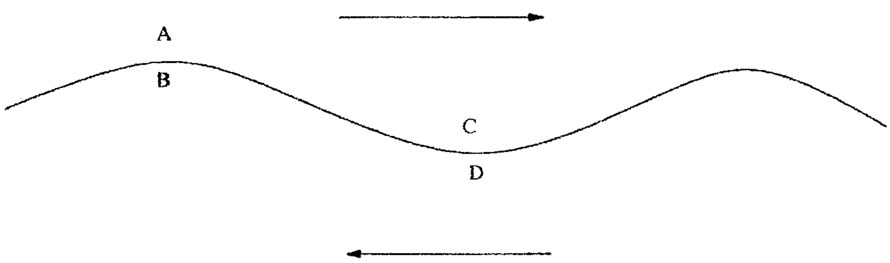
You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) It is convenient to use a reference frame that moves with a speed $U/2$ in the $+x$ direction, so that the fluid velocities in the two regions $z < 0$ and $z > 0$ appear equal and opposite. We will consider the development of a wavy (sinusoidal) perturbation to the vortex sheet. By symmetry such a disturbance does not travel in x direction. In the upper-half plane ($z > 0$), consider two streamlines, one at a sufficiently large distance (large z) where the streamline is nearly a straight line and the other close to the perturbed vortex sheet with the same sinusoidal shape. The space between these two streamlines forms a stream tube. For an incompressible flow, conservation of mass requires the flow speed to increase at crests and decrease at troughs. Through Bernoulli's equation, pressure at A is lower than at C. Similarly for the lower-half plane, the pressure at B is higher than at D. For an undisturbed flow the pressure would be a constant along the interface. Thus, it is clear that, the pressure at A tends to be lower than at B, and the fluid must accelerate upwards at A and B to balance this pressure difference. In a similar way, there is acceleration downwards at C and D. Such accelerations increase the amplitude of the perturbation and the flow is unstable, the instability being known as the Kelvin-Helmholtz instability.



Physical mechanism of Kelvin-Helmholtz instability

(b) (i) If we substitute $\phi(x,t) = Ux + f(z)\exp(st + ikx)$ in Laplace's equation, we get:

$$-k^2 f + \frac{d^2 f}{dz^2} = 0. \quad (1)$$

The general solution to this ODE is

$$f(z) = A_1 e^{-kz} + A_2 e^{kz}, \quad (2)$$

for constants A_1 and A_2. But, since the velocity must remain finite as $z \to \infty$, A_2=0, and we have

$$\phi(x,t) = Ux + A_1 e^{st+ikx-kz} \quad (3)$$

for $z > \eta(x,t)$. A similar argument, using the finiteness of velocity as $z \to -\infty$, leads to

$$\phi(x,t) = B_1 e^{st+ikx+kz} \tag{4}$$

for $z < \eta(x,t)$.

(ii) The z-coordinate of the fluid particles that make up the interface is $\eta(x,t)$. The z-velocity is therefore $D\eta/Dt$, where D/Dt is the material derivative. The velocity can also be written as $\partial\phi/\partial z$. Therefore,

$$\frac{\partial\phi}{\partial z} = \frac{D\eta}{Dt} = \frac{\partial\eta}{\partial t} + \frac{\partial\phi}{\partial x}\frac{\partial\eta}{\partial x} \tag{5}$$

on $z = \eta(x,t)$. As we are considering small amplitude disturbances, A_1, B_1 and η are all so small that their products can be neglected. This means that, after noting the form for $\phi(x,t)$ in Eq. (3), we can simplify Eq. (5) to

$$\frac{\partial\phi}{\partial z} = \frac{\partial\eta}{\partial t} + U\frac{\partial\eta}{\partial x} \tag{6}$$

on $z = \eta(x,t)$. After substituting for $\eta(x,t)$ and $\phi(x,t)$, we obtain

$$-A_1 k e^{-k\eta} = (s + ikU)\eta_0 \tag{7}$$

Linearisation simplifies this still further. For small $k\eta$ the expansion

$$e^{-k\eta} = 1 - k\eta + \cdots \tag{8}$$

shows that, when all nonlinear terms are neglected, the kinematic boundary condition simplifies to

$$A_1 = -(s/k + iU)\eta_0. \tag{9}$$

for $z > \eta(x,t)$. A similar argument for $\phi(x,t)$ from Eq. (4) leads to

$$B = s/k \tag{10}$$

for $z < \eta(x,t)$.

(iii) Applying Bernoulli's equation in $z > \eta(x,t)$, we get

$$p + \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left(\frac{\partial\phi}{\partial x}\right)^2 + \rho g\eta = p_\infty + \frac{1}{2}\rho U^2 \tag{11}$$

The form for ϕ from Eq. (3) shows that on $z = \eta(x,t)$,

$$\left(\frac{\partial\phi}{\partial x}\right)^2 = U^2 + 2UikA_1 e^{st+ikx-k\eta} + \text{nonlinear terms} \tag{12}$$

(TURN OVER for continuation of Question 1

Substitution for ϕ and η in Eq. (11) leads to

$$p(x,\eta,t) = p_\infty - \rho(s+ikU)A_1 e^{st+ikx-k\eta} - \rho g\eta_0 e^{st+ikx} \quad (13)$$

Finally, after using the expansion in Eq. (8) and neglecting any nonlinear terms, we obtain

$$p(x,\eta,t) = p_\infty - \rho(s+ikU)A_1 e^{st+ikx} - \rho g\eta_0 e^{st+ikx} \quad (14)$$

A similar consideration of the region $z < \eta(x,t)$, with ϕ from Eq. (4), leads to

$$p(x,\eta,t) = p_\infty - \rho s B_1 e^{st+ikx} - \rho g\eta_0 e^{st+ikx} \quad (15)$$

Since these two forms for the pressure must be equal

$$A_1(s+ikU) = B_1 s \quad (16)$$

Substitution for A_1 and B_1 derived in part (ii) leads to

$$(s+ikU)^2 + s^2 = 0 \quad (17)$$

The solutions of this quadratic equations are

$$s = -\frac{1}{2}kU(i \pm 1) \quad (18)$$

Examiner's comment:

(a) This was answered well by most students. About half the students found it difficult to explain why velocity would increase above a crest (or decrease above a trough) after a vortex sheet is deformed in a sinusoidal shape.

(b) The first part of this question was answered extremely well by most students. The second part was also answered well. Some students lost marks for not explaining their reasoning, particularly for dropping higher-order nonlinear terms. For the last part, about a quarter of the students did not use the correct form of the Bernoulli's equation, even though it is in the Data Card.

2 (a) Consider two thin coaxial rings of fluid A and B. Each has mass δm. The idea is to investigate the change in kinetic energy when these two rings swap positions. Initially the fluid in ring A is at radius r_1 and so has azimuthal velocity $V_1 = V(r_1)$, while ring B is at r_2 ($r_2 > r_1$) with flow speed $V_2 = V(r_2)$. The initial kinetic energy of the fluid in these two rings is therefore $1/2\delta m(V_1^2 + V_2^2)$. Now suppose that the fluid in ring A moves to radius r_2, and by continuity the fluid in ring B moves to r_1. As the fluid in ring A is moved, its angular momentum is conserved and so rV remains constant and equal to $r_1 V_1$. The velocity of the fluid A at its new radius r_2 is therefore $r_1 V_1/r_2$. By a similar argument the velocity of the fluid in B at its new radius r_1 is $r_2 V_2/r_1$. The new kinetic energy is therefore

$$\frac{1}{2}\delta m\left(\frac{r_1^2 V_1^2}{r_2^2} + \frac{r_2^2 V_2^2}{r_1^2}\right). \tag{19}$$

Hence the change is kinetic energy,

$$\Delta KE = \frac{1}{2}\delta m(r_1^2 V_1^2 - r_2^2 V_2^2)\left(\frac{1}{r_2^2} - \frac{1}{r_1^2}\right) \tag{20}$$

The circulation $\Gamma(r)$ around a circle of radius r is given by $\Gamma(r) = 2\pi r V(r)$. Therefore,

$$\Delta KE = -\frac{1}{8\pi^2}\delta m(\Gamma_1^2 - \Gamma_2^2)\left[\frac{1}{r_1^2} - \frac{1}{r_2^2}\right] \tag{21}$$

Since $r_2 > r_1$, the terms in square brackets is positive. When Γ^2 decreases with r, the perturbation releases kinetic energy from the basic flow, leading to instability. When Γ^2 increases with r, the perturbation increases the kinetic energy of the basic flow. Therefore, an energy input is required to produce such a disturbance, and we can say that the flow is stable.

(b) (i) Continuity equation is given by

$$\nabla \cdot \boldsymbol{u} = 0, \tag{22}$$

where $\boldsymbol{u} = \{U, V\}$. Because the velocity field is axi-symmetric, the continuity equation becomes

$$\frac{\partial U(r)}{\partial r} + \frac{U(r)}{r} = 0 \tag{23}$$

$$\Rightarrow \ln U = -\ln r + \ln\lambda \tag{24}$$

where λ is a constant. Therefore,

$$u = \frac{\lambda}{r} \tag{25}$$

(ii) Because the flow is assumed to be inviscid, the angular momentum would be conserved. Therefore,

$$V(r)r = \text{constant} = \beta \tag{26}$$

This implies that

$$V = \frac{\beta}{r} \tag{27}$$

(iii) $\Gamma = 2\pi Vr = 2\pi\beta = \text{constant}$. Therefore, the flow is neutrally stable.

(c) Let the rotation rate of the inner cylinder by given by Ω_1. Below a critical value of the rotation rate of the inner cylinder, say Ω_{c1}, the flow would be stable and would have the form of a basic Couette flow. At $\Omega_1 = \Omega_{c1}$, we get a new steady flow that consists of toroidal vortices, called Taylor vortices, that fill the gap between the two cylinders. The Taylor vortices are stable for a range of rotation rates $\Omega_1 > \Omega_{c1}$. But, as Ω_1 increases beyond a second critical rotation rate Ω_{c2}, the vortices become unsteady and waves propagate azimuthally around the vortices. As the rotation rate is increased still further ($\Omega_1 > \Omega_{c3}$), the flow becomes chaotic. Finally, above a fourth critical rotation rate, the flow becomes fully turbulent and the azimuthal waves vanish.

(a) This was answered well by most students
(b) The first part of the question was answered well. For the second part, about a third of the students could not solve it as they did not realize that the angular momentum is conserved if the flow is inviscid. This was unexpected as they used this concept to answer part (a). The last part of the question was answered well by most students. About a quarter of the students lost marks because they incorrectly used the total velocity instead of the tangential velocity to calculate the circulation around a circle.
(c) This part was answered extremely well by most students. Some students lost marks for not mentioning that at low rotational rates of the inner cylinder, the flow is stable and laminar.

3

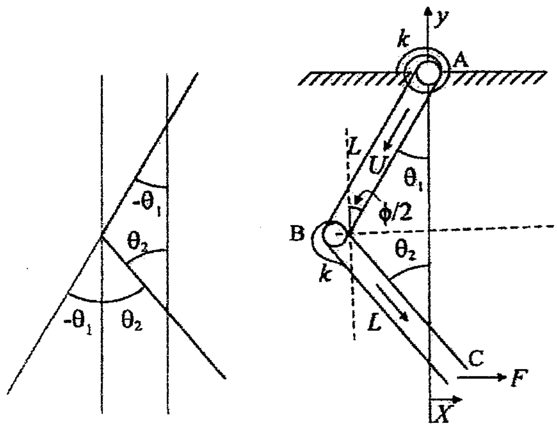


Fig. 1

(a) At point **B**, the fluid changes angle by $\theta_2+\theta_1$ anticlockwise, which we shall call ϕ. The rate of change in momentum in the direction of the bisector of **ABC** (the nearly horizontal dashed line) is $2\dot{m}U\sin(\phi/2)$. This equals the centrifugal force at point **B**. For small θ_1 and θ_2, this force equals $\dot{m}U(\theta_2+\theta_1)$.

(b) The moment about **B** due to the force F equals $FL\cos\theta_2$ (anticlockwise). The moment about **B** due to the spring at **B** equals $k(\theta_2+\theta_1)$ (clockwise). In static equilibrium, the moments balance so $FL\cos\theta_2 = k(\theta_2+\theta_1)$. For small θ_2, this becomes

$$FL = k(\theta_2+\theta_1) \qquad (1)$$

(c) For small θ_1 and θ_2, the moment about **A** due to the centrifugal force at **B** is $\dot{m}UL(\theta_2+\theta_1)$ (clockwise). For small θ_1 and θ_2, the moment about **A** due to the force F equals $2FL$ (anticlockwise). The moment about **A** due to the spring at **A** equals $k\theta_1$ (anticlockwise). In static equilibrium, the moments balance so

$$\dot{m}UL(\theta_2+\theta_1) = 2FL + k\theta_1 \qquad (2)$$

(d) The displacement, X, is equal to $L\sin(\theta_2) - L\sin(\theta_1)$. For small θ_1 and θ_2 we have

$$X = L(\theta_2-\theta_1) \qquad (3)$$

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Substituting (1) into (2) gives

$$k\theta_1 = 2FL - \dot{m}UL^2F/k \qquad (4)$$

Substituting (4) into (1) gives

$$FL = k\theta_2 - 2FL + \dot{m}UL^2F/k \qquad (5)$$

$$\Rightarrow k\theta_2 = 3FL - \dot{m}UL^2F/k \qquad (6)$$

Substituting (4) and (5) into (3) gives:

$$X = \frac{FL^2}{k}(5 - 2\dot{m}UL/k) \qquad (7)$$

(e) For $0 \leq \dot{m}U < 5k/(2L)$, the force, F, causes a displacement, X, in the same direction as the force. This is the normal behaviour one would expect from a spring. For $\dot{m}U > 5k/(2L)$, however, the force causes a displacement in the opposite direction to the force. This is very curious behaviour. It is like a spring with negative stiffness.

(f) If **C** is allowed to move vertically but not horizontally, the apparatus will buckle when $\dot{m}U > 5k/(2L)$.

This apparatus was first reported in *Nature* Vol. 296, p 135.

Examiner's comment:

(a) This part of the question was well answered by most candidates. Some candidates lost marks for not explaining their reasoning.
(b & c) Around one third of the candidates answered these sections correctly, which was a surprisingly low number. The most common mistakes were to say that the angle at **B** is θ_2, rather than $(\theta_2 + \theta_1)$ and that the distance from **A** to **C** is L rather than $2L$.
(d) No candidates answered this part correctly, even when wrong answers to (b & c) were taken into account. Most candidates got stuck because they did not try to express X in terms of θ_2 and θ_1. Many tried to use virtual work instead.
(e) There were some reasonable qualitative descriptions of the forces in the experiment but, with no correct answers to (d), there were no correct answers to (e).
(f) This part was extremely well answered. Most candidates realised that it is equivalent to a buckling pipe, which is described in the notes.

4 As part of a drying process, a membrane is held in tension T between two streams of air moving at speeds U_1 and U_2 in opposite directions to each other.

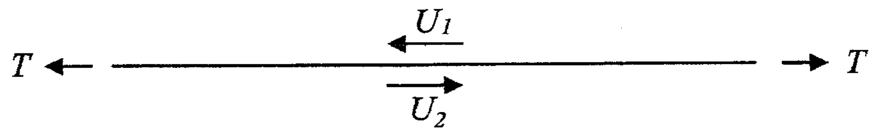

Fig. 2

Considering disturbances to the membrane with amplitude proportional to $e^{i(kx-\omega t)}$, the dispersion relation for this system is found to be:

$$(U_1+c)^2+(U_2-c)^2-kT=0 \tag{8}$$

where c, the phase speed, is equal to ω/k. The group speed is equal to $d\omega/dk$. If the system is absolutely unstable then the wave with zero group speed has positive growth rate.

(a) The amplitude, A, can be expressed as

$$A=A_0e^{i(kx-\omega t)} = e^{ikx}e^{-i\omega t}=e^{ik_rx}e^{-k_ix}e^{-i\omega_rt}e^{\omega_it} \tag{9}$$

$$= \{\cos(k_rx)+i\sin(k_rx)\}\{\cos(\omega_rt)-i\sin(\omega_rt)\}e^{-k_ix}e^{\omega_it} \tag{10}$$

From this we see that: k_r is the wavenumber in space (i.e. $k=2\pi/\lambda$, where λ is the wavelength); ω_r is the angular frequency in time (i.e. $\omega=2\pi f$, where f is the frequency in Hz); k_i is the growth/decay rate in space (i.e. for $k_i>0$, the amplitude reduces exponentially as x increases); ω_i is the growth/decay rate in time (i.e. for $\omega_i>0$, the amplitude increases exponentially as t increases).

(b) Solve (8) for ω as a function of k:

$$(U_1+c)^2+(U_2-c)^2-kT = 0 \tag{11}$$

$$\Rightarrow (U_1^2+2U_1c+c^2)+(U_2^2-2U_2c+c^2)-kT = 0 \tag{12}$$

$$\Rightarrow c^2+(U_1-U_2)c+(U_1^2+U_2^2-kT)/2 = 0 \tag{13}$$

$$\Rightarrow c = -\frac{(U_1-U_2)}{2}\pm\frac{1}{2}\sqrt{(U_1-U_2)^2-2(U_1^2+U_2^2-kT)} \tag{14}$$

$$\Rightarrow c = \frac{(U_2-U_1)}{2}\pm\sqrt{-U_1^2-2U_1U_2-U_2^2+2kT} \tag{15}$$

$$\Rightarrow c = \frac{(U_2-U_1)}{2}\pm\frac{1}{2}\sqrt{2kT-(U_1+U_2)^2} \tag{16}$$

For real k, (16) has real solutions for c when $2kT>(U_1+U_2)^2$. Therefore all disturbances are stable when $k>(U_1+U_2)^2/(2T)$.

(TURN OVER for continuation of Question 4

(c)

$$\frac{\mathrm{d}}{\mathrm{d}k}\left((U_1+c)^2+(U_2-c)^2-kT\right) = 0 \quad (17)$$

$$\left(2(U_1+c)\frac{\mathrm{d}c}{\mathrm{d}k}-2(U_2-c)\frac{\mathrm{d}c}{\mathrm{d}k}-T\right) = 0 \quad (18)$$

$$2(U_1-U_2+2c)\frac{\mathrm{d}c}{\mathrm{d}k}=T \quad (19)$$

but

$$\frac{\mathrm{d}c}{\mathrm{d}k} = \frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{\omega}{k}\right) \quad (20)$$

$$= \frac{1}{k}\left(\frac{\mathrm{d}\omega}{\mathrm{d}k}-\frac{\omega}{k}\right) \quad (21)$$

$$= \frac{c_g-c}{k} \quad (22)$$

therefore

$$2(U_1-U_2+2c)(c_g-c) = kT \quad (23)$$

therefore, the phase speed, c, equals the group speed, c_g, only when $k=0$ or $T=0$. The first condition, $k=0$, corresponds to waves with infinitely long wavelength, which is a perturbation to the base flow rather than a wave. The second condition, $T=0$, corresponds to the case with no tension, which is a very special case. In general, therefore, the phase speed differs from the group speed.

(d) We can start from either the expression derived in part (b) or that in part (c). That in (c) is more convenient.

$$2(U_1-U_2+2c)(c_g-c) = kT \quad (24)$$

$$\Rightarrow 4c(c_g-c) = kT \quad (25)$$

$$\Rightarrow c_g = c+\frac{kT}{4c} \quad (26)$$

$$(27)$$

This equals zero when $4c^2=-kT$, but we need to find an expression for c^2 in terms of k. This is easiest to do by starting from (8):

$$(U+c)^2+(U-c)^2 = kT \quad (28)$$

$$\Rightarrow (U^2+2Uc+c^2)+(U^2-2Uc+c^2)^2 = kT \quad (29)$$

$$\Rightarrow (2U^2+2c^2) = kT \quad (30)$$

$$\Rightarrow 2c^2 = kT-2U^2 \quad (31)$$

and therefore $c_g=0$ when

$$2(kT-2U^2) = -kT \quad (32)$$

$$\Rightarrow kT = 2U^2/3 \quad (33)$$

We can see from part (b) that, at this value of k, the term in the square root is $4U^2/3 - 4U^2$, which is negative, and therefore that the system is absolutely unstable because the wave with zero group velocity has positive growth rate. The implications are that small perturbations will grow on the membrane and remain in the same place. This is likely to cause corrugations in the membrane, which is a common problem in papermaking.

This problem is a simplified version of problems in papermaking described in Ann. Rev. Fluid Mech (2011) Vol. 43 pp 195–217.

Examiner's comment: This question had a lot of algebra and most candidates made some small mistakes. These were not heavily penalised and the majority of marks were given for showing physical understanding.

(a) This was answered well by most candidates.
(b) Almost all candidates knew how to determine the wavenumber above which all disturbances are stable, by setting the wavenumber to be real and searching for conditions at which c (or omega) is purely real. Most candidates scored well on this section, although many made mistakes in the algebra.
(c) Around one third of the candidates completed the algebra for this question and discovered that either the wavenumber or the tension must be zero. These candidates commented on the physical meaning of one or the other, but nobody commented on both, which was surprising.
(d) Around one quarter of the candidates completed this section. The discussion of the implementation was well answered by those who made it this far.