ET IIB

4A12 CRIB, 2011

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TURBULENCE AND VORTEX DYNAMICS

(1) (a) For a steady, high-Re, 2D flow with closed streamlines, ω = constant outside the boundary layers.

(b) (i) $\underset{\sim}{u} \cdot \nabla T = \alpha \nabla^2 T$

If $\alpha = 0$, $\underset{\sim}{u} \cdot \nabla T = 0 \Rightarrow \underset{\sim}{u} \perp^{ar}$ to ∇T

Thus T constant along the streamlines

$$\Rightarrow T = T(\psi)$$

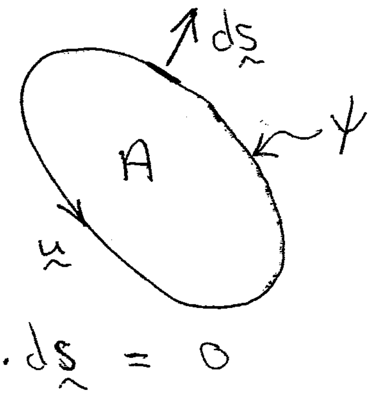
(ii) If α small, $T = T(\psi)$ + (small correction)

$$\Rightarrow \quad \underset{\sim}{u} \cdot \nabla T = \alpha \nabla \cdot [\nabla T]$$

$$= \alpha \nabla \cdot [\nabla T(\psi) + (\text{small correction})]$$

$$= \alpha \nabla \cdot [T'(\psi) \nabla \psi + (\text{small correction})]$$

(iii) Ignoring small correction



$$\int \underset{\sim}{u} \cdot \nabla T \, dA = \alpha \int \nabla \cdot [T'(\psi) \nabla \psi] \, dA$$

Apply Gauss:

$$\int \underset{\sim}{u} \cdot \nabla T \, dA = \int \nabla \cdot [T \underset{\sim}{u}] \, dA = \oint T \underset{\sim}{u} \cdot d\underset{\sim}{s} = 0$$

$$\int \nabla \cdot [T'(\psi) \nabla \psi] \, dA = \oint T'(\psi) \nabla \psi \cdot d\underset{\sim}{s} = T'(\psi) \oint \nabla \psi \cdot d\underset{\sim}{s}$$

(T' is constant on surface)

Thus $\underline{\underline{\alpha T'(\psi) \oint \nabla \psi \cdot d\underset{\sim}{s} = 0}}$

(b) (iv) $\int \nabla\psi \cdot d\underset{\sim}{s} = -\oint \underset{\sim}{u}\cdot d\underset{\sim}{r} \Rightarrow \alpha T'(\psi) \oint \underset{\sim}{u}\cdot d\underset{\sim}{r} = 0$

But $\alpha \neq 0$, $\oint \underset{\sim}{u}\cdot d\underset{\sim}{r} \neq 0 \Rightarrow \underline{T'(\psi) = 0}$

Physical interpretation: T eventually evens out to a a constant value by slow cross-stream diffusion.

(c) In a similar way,

$$\underset{\sim}{u}\cdot\nabla\omega = \nu\nabla^2\omega \Rightarrow \omega = \omega(\psi) \text{ if } \nu \text{ small}$$

$$\Rightarrow \underset{\sim}{u}\cdot\nabla\omega \approx \nu\nabla\cdot[\omega'(\psi)\nabla\psi]$$

$$\Rightarrow \nabla\cdot[\omega\underset{\sim}{u}] \approx \nu\nabla\cdot[\omega'(\psi)\nabla\psi]$$

Integrate: $\underbrace{\int \omega\underset{\sim}{u}\cdot d\underset{\sim}{s}}_{0} = \nu\omega'(\psi)\int\nabla\psi\cdot d\underset{\sim}{s} = -\nu\omega'(\psi)\oint\underset{\sim}{u}\cdot d$

$$\Rightarrow \nu\omega'(\psi)\oint\underset{\sim}{u}\cdot d\underset{\sim}{r} = 0 \Rightarrow \underline{\underline{\omega'(\psi) = 0}}.$$

Thus ω uniform over the flow.

(d) As for T, slow cross-stream diffusion eradicates any gradients in ω.

Examiner's comment:

Q1. A very popular question on the Prandtl-Batchelor theorem with 43 attempts out of 44. The answers were very good by and large, but one or two students seemed quite lost.

(2) (a) Kelvin: for inviscid fluid $\frac{D}{Dt}\oint \underset{\sim}{u}\cdot d\underset{\sim}{r} = 0$

Helmholtz # 1: Vortex lines frozen into an inviscid fluid, like dye lines

Helmholtz # 2: In an inviscid fluid the flux $\Phi = \int \underset{\sim}{\omega}\cdot d\underset{\sim}{S}$ is constant along a vortex tube and independent of time.

(b) (i) $d\underset{\sim}{r} = \lambda\, \underset{\sim}{\omega}(\underset{\sim}{x}_A, t=0)$, $\frac{D\lambda}{Dt} = 0$

$$\frac{D\underset{\sim}{c}}{Dt} = \frac{D}{Dt} d\underset{\sim}{r} - \frac{D}{Dt}(\lambda \underset{\sim}{\omega}) = \frac{D}{Dt} d\underset{\sim}{r} - \lambda \frac{D\underset{\sim}{\omega}}{Dt}$$

$$\Rightarrow \frac{D\underset{\sim}{c}}{Dt} = (d\underset{\sim}{r}\cdot\nabla)\underset{\sim}{u} - \lambda(\underset{\sim}{\omega}\cdot\nabla)\underset{\sim}{u} = (d\underset{\sim}{r} - \lambda\underset{\sim}{\omega})\cdot\nabla\underset{\sim}{u}$$

$$\Rightarrow \underline{\underline{\frac{D\underset{\sim}{c}}{Dt} = \underset{\sim}{c}\cdot\nabla\underset{\sim}{u}}}$$

(ii) $\underset{\sim}{c}(t=0) = 0 \Rightarrow \frac{D}{Dt}\underset{\sim}{c} = 0$ at $t=0$

$\Rightarrow$ $\underset{\sim}{c}$ remains zero,

$\Rightarrow$ $d\underset{\sim}{r} = \lambda\underset{\sim}{\omega}$ at all times

$$\Rightarrow d\underset{\sim}{r} = \frac{|d\underset{\sim}{r}(t=0)|}{|\underset{\sim}{\omega}(t=0)|}\underset{\sim}{\omega}$$

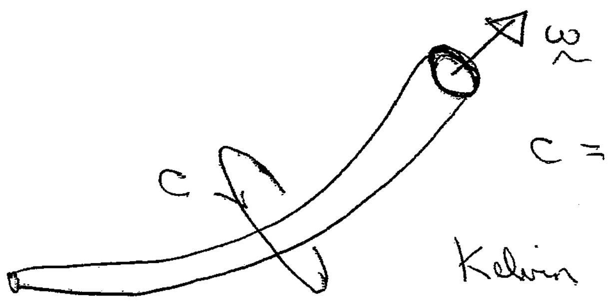
$$\Rightarrow \underline{\underline{\frac{d\underset{\sim}{r}}{|d\underset{\sim}{r}(t=0)|} = \frac{\underset{\sim}{\omega}}{|\underset{\sim}{\omega}(t=0)|}}}$$

$\Rightarrow$ $d\underset{\sim}{r}$ always remains $/\!/^{\text{el}}$ to $\underset{\sim}{\omega}$

(b) (iii) Since $d\underset{\sim}{r}$ remains $\parallel^{\text{el}}$ to $\underset{\sim}{\omega}(\underset{\sim}{x}_A)$, the two vectors $d\underset{\sim}{r}$ and $\underset{\sim}{\omega}$ must stay coincident

[Alternatively, $d\underset{\sim}{r}$ and $\underset{\sim}{\omega}$ evolve according to the <u>same</u> evolution equation, so if they start coincident they must stay coincident.]

(c)



C = material line

Kelvin $\Rightarrow \frac{D}{Dt}\oint_C \underset{\sim}{u}\cdot d\underset{\sim}{l} = 0$

$\Rightarrow \oint \underset{\sim}{u}\cdot d\underset{\sim}{l} = \int \underset{\sim}{\omega}\cdot d\underset{\sim}{S} = \text{constant}$

Thus flux through C does not change so C always encloses the vortex tube. This is true of <u>any</u> material line which encircles the tube at $t=0$.

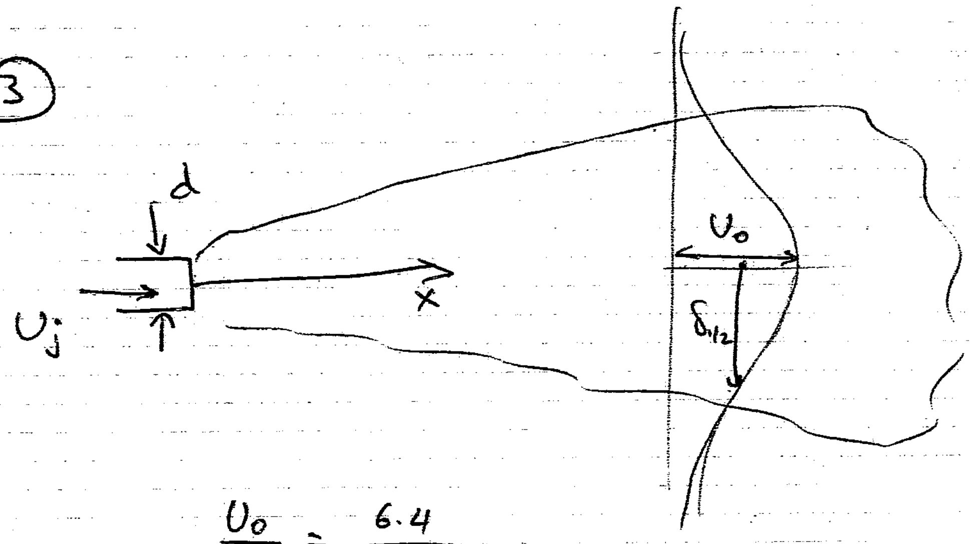
Thus $\underset{\sim}{\omega}$ – tube must also move with the fluid.

Examiner's comments:

Q2. A very popular question on Helmholtz's laws of vortex dynamics, with 43 attempts out of 44. The answers were uniformly good.

4A12 2011

(3)



$$\frac{U_0}{U_j} = \frac{6.4}{x/d}$$

(a) The half-width of the jet $\delta_{1/2}$ can be used to estimate the integral lengthscale L.

For a jet, $\delta_{1/2} \simeq 0.1\,x$ (NOTE: the thickness of all thin shear flows $\ll x$, so $\delta \sim 10\% x$ is a good estimate for all such flows).

The self-preserving jet has $\frac{u}{U_0}$ = constant (u : turbulent velocity) and we know that $\frac{u}{U_0} \sim 25\%$ for a jet.

Therefore, the various turbulence scales are:

$$T = \frac{L}{u} = \frac{0.1\,x}{0.25\,U_0} = \frac{0.1}{0.25 \cdot 6.4} \frac{x}{U_j\,(x/d)^{-1}}$$

$$\Leftrightarrow \quad T = \frac{0.1}{0.25 \cdot 6.4} \frac{d}{U_j} \left(\frac{x}{d}\right)^2$$

(~~integral~~ eddy turnover time)

$$T' = \frac{L}{U_0} = \frac{0.1\,x}{6.4\,U_j\,(x/d)^{-1}} = \frac{0.1}{6.4} \frac{d}{U_j} \left(\frac{x}{d}\right)^2$$

(integral timescale)

3 cont'd

The Kolmogorov lengthscale η_k is $\eta_k = L\, Re_t^{-3/4}$

The turbulent Reynolds number is $Re_t = \frac{uL}{\nu}$

$$Re_t = \frac{uL}{\nu} = \frac{(0.25\,U_o)(0.1\,x)}{\nu} = (0.25 \cdot 6.4 \cdot 0.1)\ \frac{U_j\, d}{\nu}$$

$$= 0.16\ \frac{U_j\, d}{\nu}$$

The fastest frequency that the hot wire must follow is given by the need to resolve the smallest eddy (η_k) as it is convected by the mean flow past the probe. Therefore, the fastest frequency is

$$\frac{U_o}{\eta_k} = \frac{6.4\ U_j \left(\frac{x}{d}\right)^{-1}}{(0.1\ x)\left(0.16\ U_j\, d/\nu\right)^{-3/4}}$$

Putting numbers; for $d = 10$ mm, $U_j = 50$ m/s, ν for air at 1 bar & 300K $\sim 1.5\times10^{-5}$ m²/s, we get:

$Re_t = 53333$, $L = 0.05$ m, $U_o = 6.4$ m/s

$\eta_k = 0.08$ mm $\Rightarrow$ $\boxed{\frac{U_o}{\eta_k} \approx 80 \text{ kHz}}$

(b) The statistical convergence of our measured mean follows

$$(\text{relative error}) \sim \frac{u}{U_o}\left(\frac{T'}{T_{samp}}\right)^{1/2}$$

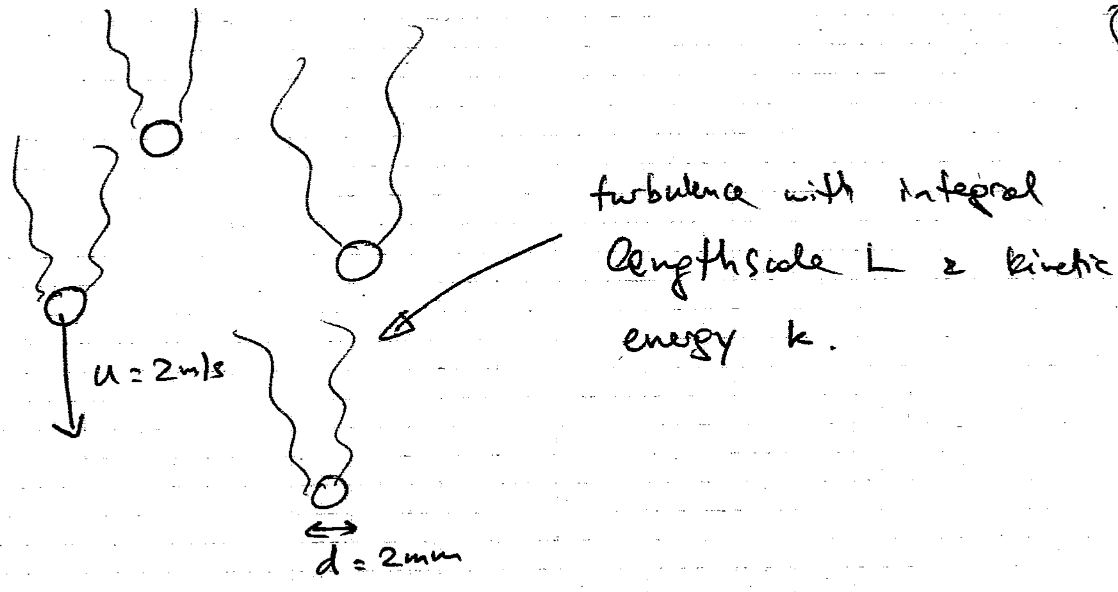
. Therefore, the sampling time T_{samp} must be $625\ T'$ for 1% relative error. $\left(\left(\frac{0.25}{0.01}\right)^2\right)$

$\Rightarrow$ $\underline{\underline{4.9\ s}}$, NOTE the use of T', not the eddy turnover time T, due

Examiner's comment:

Q3. A straight forward question on turbulent jets which was avoided by almost everyone, with only 7 attempts out of 44. . This is the second year running that the question on turbulent jets was avoided by nearly all candidates

(4)



Average distance between drops = $n^{-1/3}$ = 0.046m

(a) We may expect the lengthscale of the turbulence to be larger than d, since the scale of the turbulence inside the wake (which scales with d) will grow as the wakes evolve.
We may expect the growth mechanism to stop when the wake is intercepted by a second drop, which will happen at the average distance between the drops ($n^{-1/3}$).

(b) From the k-equation (data card) for stationary homogeneous systems, the body force is the only production mechanism.

$$\overline{g_i' u_i} = \varepsilon$$

For our problem ~~this~~ the body force on the fluid per unit mass is needed. Since the drops fall at their terminal velocity, the drag = m.g (m = mass of drop).

$$\Rightarrow \underbrace{\underbrace{\left(\rho_L \frac{\pi}{6} d^3\right) \cdot g \cdot n}_{\text{force per unit fluid volume}} / \rho}_{\text{force per unit}} \cdot \underset{\substack{\uparrow \\ \text{terminal velocity}}}{u} = \text{power in per unit mass}$$

4 cont'd

$$\therefore \quad \frac{\rho_L}{\rho} \frac{\pi}{6} d^3 g n u = \varepsilon$$

$$= \frac{k^{3/2}}{L} \quad \text{(using the usual model for } \varepsilon\text{)}$$

$$\Rightarrow \quad \frac{\rho_L}{\rho} \frac{\pi}{6} d^3 g n \; u \; L = k^{3/2}$$

So, for $L = d$, $k^{1/2} = 0.11$ m/s

and for $L = n^{-1/3}$, $k^{1/2} = 0.33$ m/s

($\rho_L = 1000$ kg/m^3
$\rho = 1.1$ kg/m^3
$g = 9.81$ m^2/s
$d = 2$ mm
$n = 10^4/$m^3
$u = 2$ m/s)

For $k^{1/2} = u$, $L = \dfrac{u^2}{\left(\frac{\rho_L}{\rho} \frac{\pi}{6} d^3 g n\right)}$

$= 10.71$ m huge!

Clearly, $k^{1/2} \sim u$ is an overestimate.

(c) The wake's growth will be disrupted at an average distance $x \sim n^{-1/3} = 0.046$ m i.e. $23d$ from the drop. At $\frac{x}{d} = 23$, the wake thickness will be $\delta_{1/2} = d \cdot 0.6 \, (23)^{1/3} = 3.41$ mm.

This actually begins to make sense: greater than d, not as large as $n^{-1/3}$. The truth is probably somewhere around here.

The turbulence inside the wake will be a fraction of the outer velocity u, so our estimates in (b) make some sense too.

Examiner's comment:

Q4. A popular question on turbulent energy dissipation with 39 attempts out of 44. The attempts were reasonably satisfactory, but very few candidates could estimate the energy input to the turbulence.