PROF D CEBON
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Part IIb, Paper 4C8 Applications of Dynamics 2011

1.(a) Bookwork - see lecture notes.

(b) $m(\dot{v} + u\Omega) + 2c\frac{v}{u} + (a-b)c\frac{\Omega}{u} = c\delta$

$I\dot{\Omega} + c(a-b)\frac{v}{u} + c(a^2+b^2)\frac{\Omega}{u} = ac\delta$

putting $\delta = -K\Omega$ and writing in matrix form:

$$\begin{bmatrix} m & 0 \\ 0 & I \end{bmatrix}\begin{Bmatrix} \dot{v} \\ \dot{\Omega} \end{Bmatrix} + \begin{bmatrix} \frac{2c}{u} & mu + \frac{(a-b)c}{u} + Kc \\ \frac{c(a-b)}{u} & \frac{c(a^2+b^2)}{u} + aKc \end{bmatrix}\begin{Bmatrix} v \\ \Omega \end{Bmatrix} = \underline{0}$$

For stability, put $v = V_0 e^{st}$, $\Omega = \Omega_0 e^{st}$ & solve the characteristic eqn:

$$\Rightarrow \begin{vmatrix} mus + 2c & mu^2 + (a-b)c + Kcu \\ c(a-b) & Ius + c(a^2+b^2) + aKcu \end{vmatrix} \begin{matrix} V_0 \\ \Omega_0 \end{matrix} = 0$$

$$\Rightarrow (mus+2c)(Ius + c(a^2+b^2) + aKcu) - c(a-b)(mu^2 + (a-b)c + Kcu) = 0$$

$$\Rightarrow \underbrace{(Imu^2)}_{a_2} s^2 + \underbrace{\{2cIu + mu[c(a^2+b^2) + aKcu]\}}_{a_1} s$$

$$+ \underbrace{\{2\cancel{c}[c(a^2+b^2) + aKcu] - \cancel{c}(a-b)[mu^2 + (a-b)c + Kcu]\}}_{a_0} = 0$$

c) Stability Conditions: all a's +ve

(i) a_2 is always > 0

(ii) a_1 is always > 0 as long as $K > 0$

(If $K < 0$, then $\cancel{c}(a^2+b^2) - a|K|\cancel{c}u > 0$
ie $|K| < \frac{a^2+b^2}{au}$)

(iii) $a_0 > 0$:

$$2c(a^2+b^2) + 2aKcu - (a-b)mu^2 - (a-b)^2c - (a-b)Kcu > 0$$

1 cont

ie $\quad c(a+b)^2 - (a-b)mu^2 + (a-b)Kcu > 0$

for $K=0$: either need $(a-b) < 0$ ie $b > a$ [CG fwd of middle wheelbase

or If $a > b$, then there is a critical speed at

$$u^2 = \frac{c(a+b)^2}{(a-b)m} = \frac{cl^2}{(a-b)m} \quad \text{as expected}$$

If $K > 0$, then:
vehicle is always stable if

$$mu^2(a-b) - (a-b)Kcu - c(a+b)^2 < 0$$

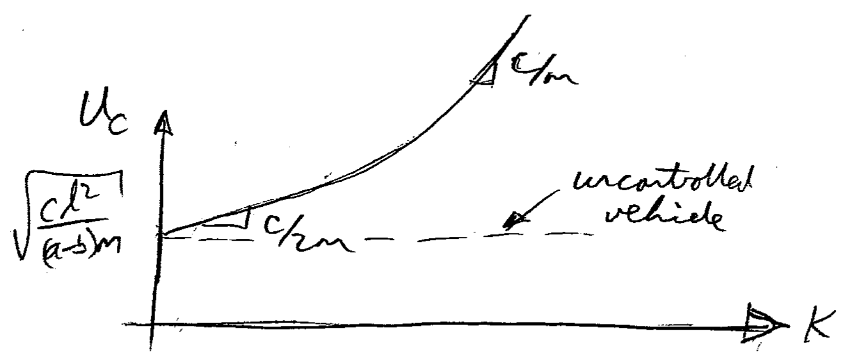
Solns are:

$$u = \frac{(a-b)Kc \pm \sqrt{(a-b)^2K^2c^2 + 4m(a-b)c(a+b)^2}}{2m(a-b)}$$

ie

$$u_c = \frac{Kc}{2m} + \sqrt{\frac{K^2c^2}{4m^2} + \frac{cl^2}{(a-b)m}}$$

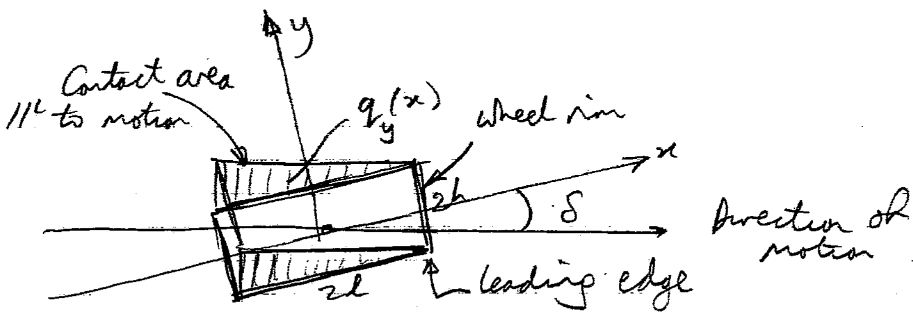
(Consider only the +ve root for forwards motion)



Considering only forwards motion K always increases U_c

Autonomous Car Stability. Attempted by all candidates. Part (a) was bookwork, generally well done. In part (b) quite a few candidates attempted a steady-state cornering analysis when asked to calculate the straight-line stability. Why?

2. (a)



Lateral deflection of tips of bristles from unstressed position is

$$q_y(x, y) = \delta(l - x)$$

Long'l deflection is zero $\quad q_x(x, y) = 0$

$$Y = \iint K_y q_y \, dA = \int_{-h}^{h} dy \int_{-l}^{l} K_y \, \delta(l-x) \, dx = 4l^2 h K_y \delta$$

Now $\quad \alpha = \frac{v}{u} - \delta \implies \alpha = -\delta$

$$\therefore \quad Y = -C_{22}\alpha \implies \underline{\underline{C_{22} = 4l^2 h K_y}}$$

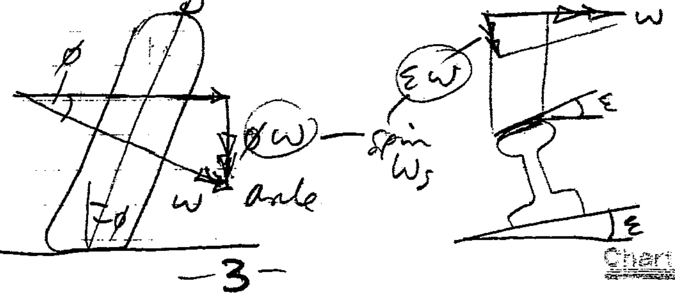
$$N = \iint (x\sigma_y - y\sigma_x) \, dA = \int_{-h}^{h} dy \int_{-l}^{l} x K_y \, \delta(l-x) \, dx = -\frac{4}{3} l^3 h K_y \delta$$

$$N = C_{32}\alpha \implies \underline{\underline{C_{32} = \frac{4}{3} l^3 h K_y}}$$

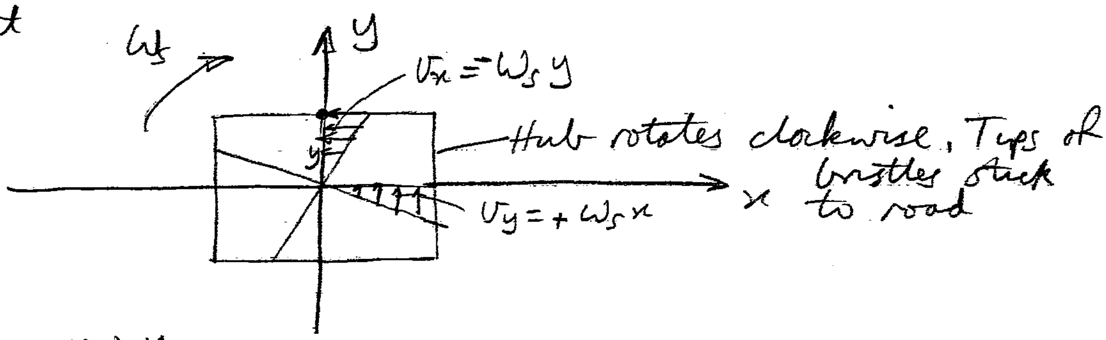
<u>Assumptions:</u>

(i) Bristles are attached to a rigid wheel rim
(ii) Tips of bristles are not connected
(iii) Bristles are elastic → no mass or damping
(iv) Bristles stick to road surface → no microslip

(b) When contact plane is not parallel to axis of rotation, the wheel has a "spin" compt of angular velocity perpendicular to contact patch — occurs in railway wheels & for cambered car tyres



2(b) cont



$v_x = -\omega_s y$

$v_y = \omega_s x$

Convection : $\dfrac{dx}{dt} = -u$ ie $dt = -dx/u$

$$q_x = \int v_x\,dt = \int -\omega_s y\left(-\frac{dx}{u}\right) = xy\Psi + f_1(y)$$

$$q_y = \int v_y\,dt = \int \omega_s x\left(-\frac{dx}{u}\right) = -\frac{x^2}{2}\Psi + f_2(y)$$

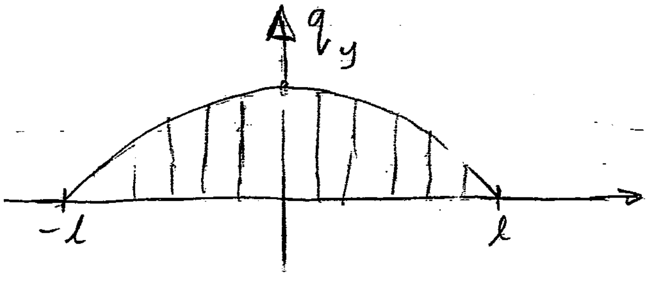
Boundary conditions $q_x(l, y) = q_y(l, y) = 0$

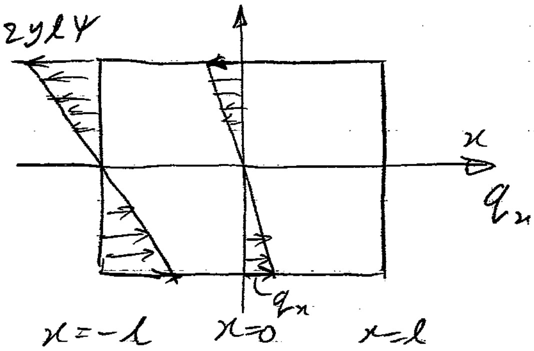
$\Rightarrow f_1(y) + ly\Psi = 0 \Rightarrow f_1 = -ly\Psi$

& $f_2 = l^2/2\,\Psi$

So $q_x(x,y) = (x-l)y\Psi$

& $q_y(x,y) = \dfrac{(l^2 - x^2)}{2}\Psi$





Lateral Y force

(No longitudinal force)

Realigning moment N

$$Y = \int_{-h}^{h} dy \int_{-l}^{l} K_y \frac{\Psi}{2}(l^2 - x^2)\,dx = \underbrace{\frac{4}{3} l^3 h K_y}_{C_{23}} \Psi$$

$$N = \int_{-h}^{h} dy \int_{-l}^{l} K_y \frac{\Psi}{2} x(l^2 - x^2) - K_x \Psi y^2 (x - l)\,dx = \underbrace{\frac{4}{3} l^2 h^3 K_x}_{C_{33}} \Psi$$

2(c) Cont

Cars (i) Neglect spin creep — small

(ii) Neglect realigning moment due to α since pneumatic trail is small relative to wheelbase

(iii) Neglect X — wheels free wheel or have differentials

$\Rightarrow \underline{\underline{Y = C\alpha}}$.

Railways (i) & (ii) as per car

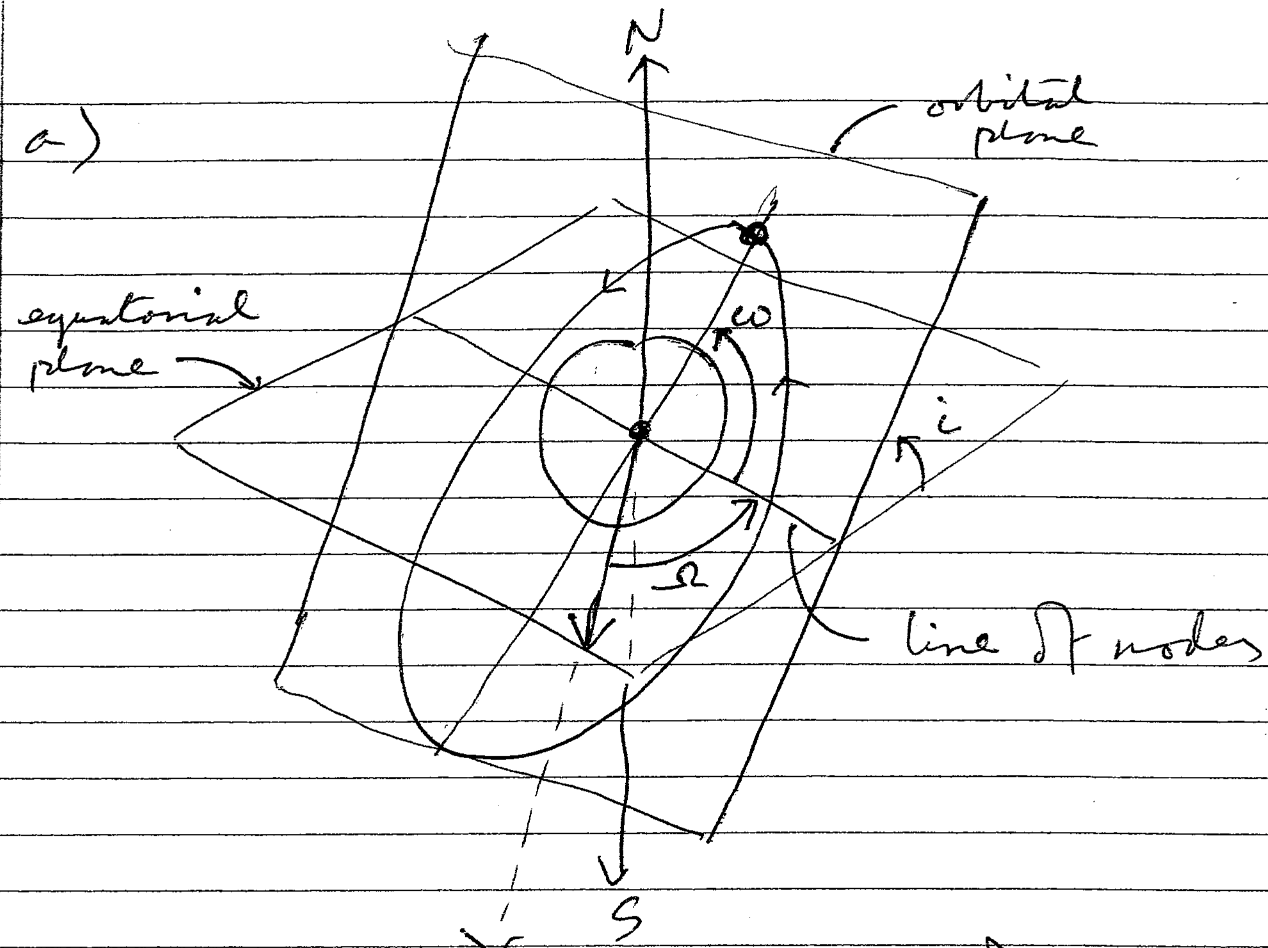
(iii) Can't ignore X because two wheels on an axle don't free wheel

(iv) Reasonable to assume $K_x = K_y$

So $Y = C\alpha$ & $X = C\xi$

Linear creep equations and their application to simple vehicle dynamics: This question was generally well done. A surprisingly large number of candidates got the wrong signs because $\alpha = -\delta$. In part (c), many students recommended the use of C_{32} in car models, while (correctly) ignoring it in Q1.

3. a)



γ — first point of aries

Parameters

Ω - right ascension of ascending node
(angle between 1st point of aries and line of nodes, ascending node

i - inclination of orbit plane w.r.t. equatorial plane

ω - argument of perigee
angle from ~~axis~~ line of nodes to line of apses

a, e - major semi axis and

3 cont

eccentricity of the ellipse.

1st point of Aries is the ascending node of the Sun's "orbit" about the Earth.

Also need the True Anomaly of the satellite, to fix its position on the orbit.

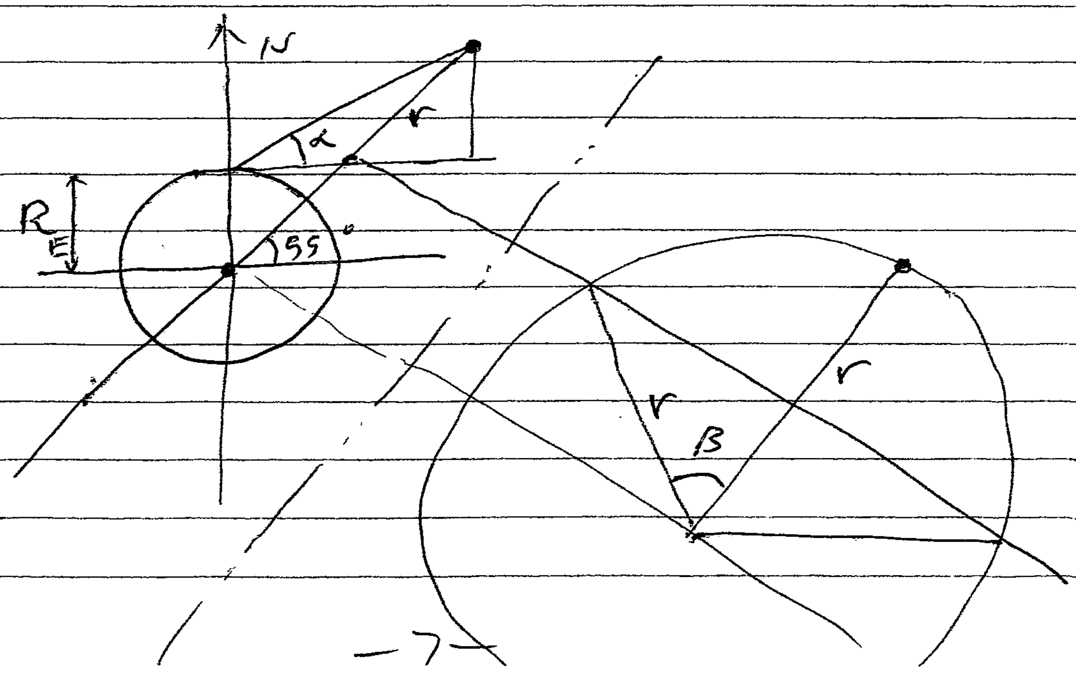
6) For a circular orbit, we can write

$$r\omega^2 = \mu / r^2 \quad \text{so} \quad r^3 = \mu / \omega^2$$

but $\omega = \frac{2\pi}{T}$, so $r = \sqrt[3]{\mu \left(\frac{T}{2\pi}\right)^2}$

$$\therefore \text{radius} = \sqrt[3]{398603\left(\frac{12 \times 3600 \times {}^{365.25}/_{366.25}}{2\pi}\right)^2}$$

$$= \underline{26,561.8 \text{ km}}$$



3 cont

i) We can write

$$\tan\alpha = \frac{r\sin 55 - R_E}{r\cos 55}$$ — 6378 km from D.B.

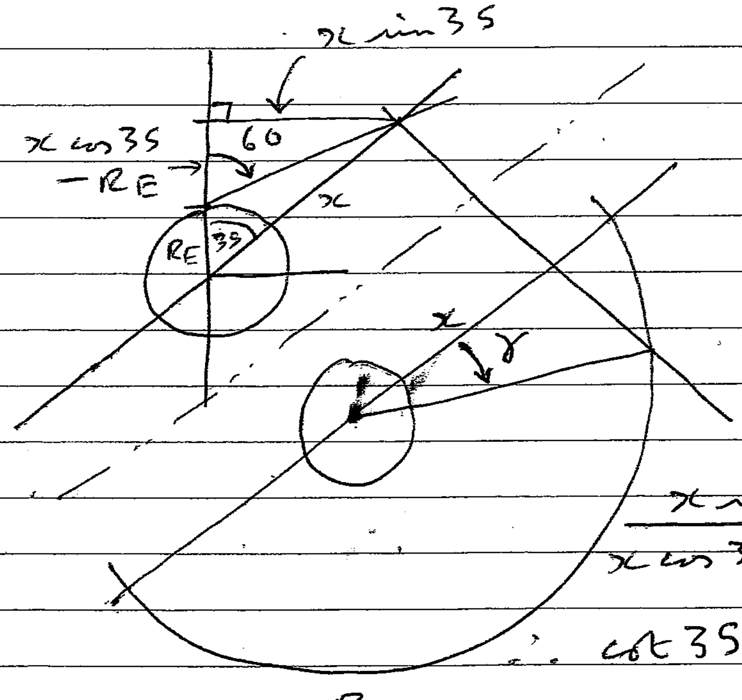
$$\rightarrow \underline{\alpha = 45.3^\circ}$$

ii) Then, $\frac{R_E}{\sin 55} = r\cos\beta$

$$\text{So } \beta = \cos^{-1}\left(\frac{R_E}{r\sin 55}\right) = 73^\circ$$

Hence orbit is visible for $\frac{73}{180}$ $\underline{= 0.4 \text{ of the time}}$

iii)



$$\frac{x\sin 35}{x\cos 35 - R_E} = \tan 60$$

$$\therefore \cot 35 - \frac{R_E}{x\sin 35} = \cot 60$$

$$\therefore x = \frac{R_E}{(\cot 35 - \cot 60)\sin 35} = 13070 \text{ km}$$

$$\text{So } \gamma = \cos^{-1}\left(\frac{13070}{26562}\right) = 60^\circ$$

So average no. of satellites visible $= \frac{60}{180} \times 24 = \underline{8}$

3 cont

c) Visibility is OK, but elevation will always be quite low - this will reduce the accuracy of height measurements, in particular - by a factor of $> \sqrt{2}$, compared to places where satellites can be seen overhead.

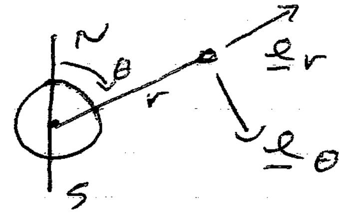
Hence expect height accuracy of ~4 centimetres, compared to ~2.5 elsewhere.

Orbit parameters and GPS performance at North Pole: This was not a particularly popular question, and few candidates were able to answer both parts well. There were several unusual descriptions of the 'First point of Aries', and some candidates had difficulty with the final section of part (b). Answers to part (c) were generally relevant and sensible.

4 a) Radial force $= \dfrac{\partial u}{\partial r}$ (outwards)

$$= -\mu/r^2 + \frac{3\mu J_2 R^2}{r^4} P_2(\cos\theta)$$

$$= \mu/r^2 + \frac{3\lambda}{r^4}\,\frac{3\cos^2\theta - 1}{2}$$



But $3\cos^2\theta - 1 = \tfrac{3}{2} + \tfrac{3}{2}\cos 2\theta - 1$

$$= \tfrac{1}{2}(1 + 3\cos 2\theta)$$

So radial force $= -\dfrac{\mu}{r^2} + \dfrac{3\lambda}{4r^4}(1 + 3\cos 2\theta)$

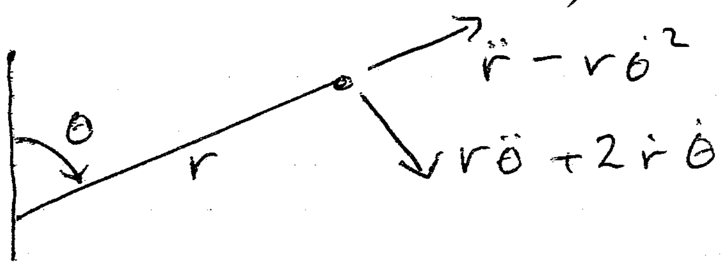
Tangential force $= \dfrac{1}{r}\dfrac{\partial u}{\partial \theta}$ in $\underline{e}_\theta$ direction

$$= \frac{\mu J_2 R^2}{r^4}\, 3\cos\theta\sin\theta$$

So tangential force $= \dfrac{3\lambda}{2r^4}\sin 2\theta$

(both per unit mass)

In polar co-ordinates, accelerations are:



Equating these to the specific forces above gives the required results.

Angular momentum does NOT stay constant, as $r\ddot{\theta} + 2\dot{r}\dot{\theta} = \frac{1}{r}\left(\frac{d}{dt}(r^2\dot{\theta})\right) \neq 0$

6) For this orbit,

$r = 870 + 6378 = 7248$ km

Fluctuating element of radial force is $\frac{3\lambda}{4r^4} 3\cos 2\theta$

Which has an amplitude of

$$\frac{9\mu J_2 R^2}{4r^4}$$

So amplitude of force is

$$\frac{9 \times 398603 \times 1082 \times 10^{-6} \times 6378^2 \times 1400}{4 \times 7248^4}$$

$$= 20.02 \times 10^{-3} \frac{\text{kg km}}{\text{s}} = \underline{20.02 \text{ N}}$$

Peak occurs twice per orbit, when $\theta = 0°$ and $180°$, i.e. at poles.

Frequency of <u>orbit</u> is given by

$$\omega^2 = \mu / r^3 \Rightarrow \omega = 1.023 \times 10^{-3} \text{ rad/s}$$

$$f = \frac{\omega}{2\pi} = 162.8 \times 10^{-6} \text{ Hz}$$

So frequency of fluctuation is

$$\underline{325.6 \times 10^{-6} \text{ Hz}}$$

Perturbations in polar orbits: A straightforward question, which was generally well answered, with the great majority of candidates correctly concluding that angular momentum is not conserved in such an orbit. The final part yielded a variety of different numerical answers, mainly because candidates failed to allow for the use of kilometres as a measure of distance in the data sheet.