PART IIB 2011 **4C9 CONTINUUM MECHANICS**

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Solutions

1.)

(a)

(i) $f_i = c_{ij} t_j$

$f_2 = c_{21} t_1 + c_{22} t_2 + c_{23} t_3$

(ii) $f_i = \varepsilon_{ijk} T_{jk}$

$f_2 = \varepsilon_{2jk} T_{jk} = \varepsilon_{213} T_{13} + \varepsilon_{231} T_{31} = -T_{13} + T_{31}$

(b) Recall $\varepsilon_{pqs} \varepsilon_{snr} = \delta_{pn} \delta_{qr} - \delta_{pr} \delta_{qn}$

Identify q with n

$\varepsilon_{pqs} \varepsilon_{sqr} = \delta_{qp} \delta_{rq} - \delta_{qq} \delta_{rp}$

$= \delta_{rp} - 3\delta_{pr} = -2\delta_{pr}$

(c)

(i) $A_{ij} = a_{pi} a_{qj} A'_{pq}$

Hence $A_{ii} = a_{pi} a_{qi} A'_{pq} = \delta_{pq} A'_{pq} = A'_{pp} = A'_{ii}$

(ii) $A_{ij} A_{ij} = a_{pi} a_{qj} A'_{pq} a_{mi} a_{nj} A'_{mn}$

$= \delta_{pm} \delta_{qn} A'_{pq} A'_{mn} = A'_{pq} A'_{pq} = A'_{ij} A'_{ij}$

(iii) $\varepsilon_{ijk}\varepsilon_{kjp}A_{ip} = \varepsilon_{ijk}\varepsilon_{kjp}a_{mi}a_{np}A'_{mn}$

$$= (\delta_{ij}\delta_{jp} - \delta_{ip}\delta_{jj})a_{mi}a_{np}A'_{mn}$$

$$= (\delta_{mn} - \delta_{mn}\delta_{jj})A'_{mn}$$

$$= (\delta_{mj}\delta_{nj} - \delta_{mn}\delta_{jj})A'_{mn}$$

$$= \varepsilon_{mjk}\varepsilon_{kjn}A'_{mn}$$

(d) $A_{ij} = A_{ji}$; $B_{ij} = -B_{ji}$

$A_{ij}B_{ij} = -A_{ji}B_{ji}$

$A_{ij}B_{ij} + A_{ji}B_{ji} = A_{ij}B_{ij} + A_{pq}B_{pq} = 0$

Since all indices are dummy

$A_{pq}B_{pq} = A_{ij}B_{ij} \Rightarrow 2A_{ij}B_{ij} = 0$

$\Rightarrow A_{ij}B_{ij} = 0$

A popular question, well-answered by most candidates. The main part the candidates found difficult was part (c) where they had to show invariance under rotations.

2)

(a) $x = r\cos\theta \;;\quad y = r\sin\theta$

$$\frac{\partial r}{\partial x} = \cos\theta \;;\quad \frac{\partial r}{\partial y} = \sin\theta$$

$$\frac{\partial \theta}{\partial x} = \frac{-y}{r^2} = -\frac{\sin\theta}{r} \;;\quad \frac{\partial \theta}{\partial y} = \frac{x}{r^2} = \frac{\cos\theta}{r}$$

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial \phi}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\frac{\cos\theta}{r}$$

(b)

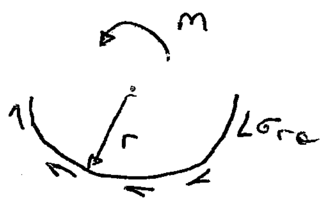
$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}$$

$$= \frac{1}{r^2}\left(-2A\sin 2\theta\right)$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = 0$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = -\frac{A}{r^2}\left(1+\cos 2\theta\right)$$

$$M = \int_{\pi/2}^{-\pi/2} \sigma_{r\theta}\, r^2\, d\theta$$



$$\Rightarrow \quad A = \frac{M}{\pi}$$

ie $\quad \sigma_{rr} = -\frac{2M}{\pi r^2}\sin 2\theta, \quad \sigma_{\theta\theta} = 0$

$$\sigma_{r\theta} = -\frac{M}{\pi r^2}(1+\cos 2\theta)$$

(c)

$$\phi_1 = -\phi(x, y+a) + \phi(x,y)$$

for $a \to 0$

$$\phi_1 = \phi(x,y) - \left[a\frac{\partial \phi}{\partial y} + \phi(x,y)\right]$$

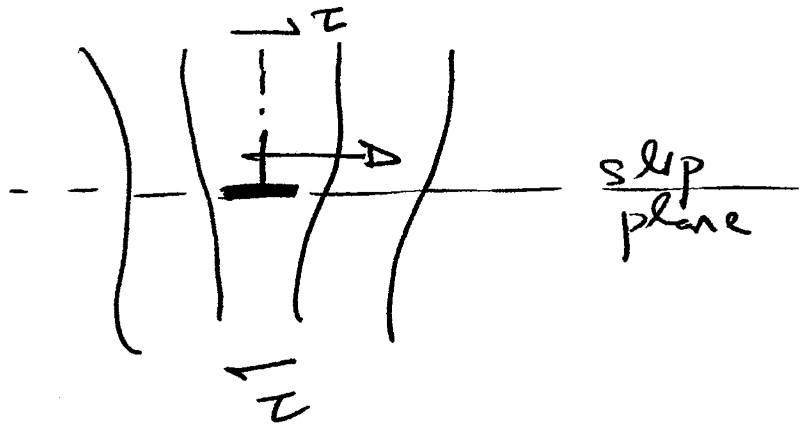
$$= -a\frac{\partial \phi}{\partial y}$$

But $\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\frac{\cos\theta}{r}$

$$\phi_1 = -\frac{2Ma}{\pi r}\cos^3\theta$$

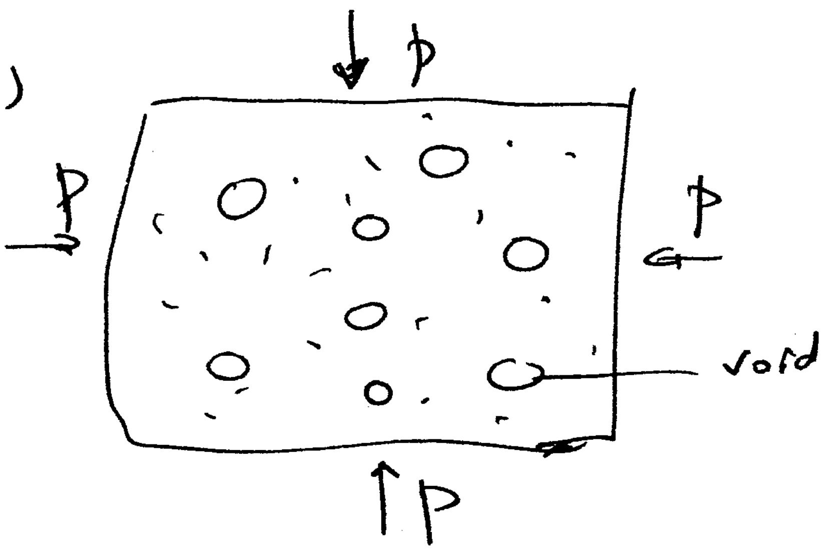
On the whole the question was poorly attempted. A number of students found it hard to calculate the moment by force equilibrium and also to use superposition in part (c).

3 (a) Plastic deformation of metals is by the motion of dislocations and/or twinning. Both involve shear without a volume change.



motion of an edge dislocation on a slip plane

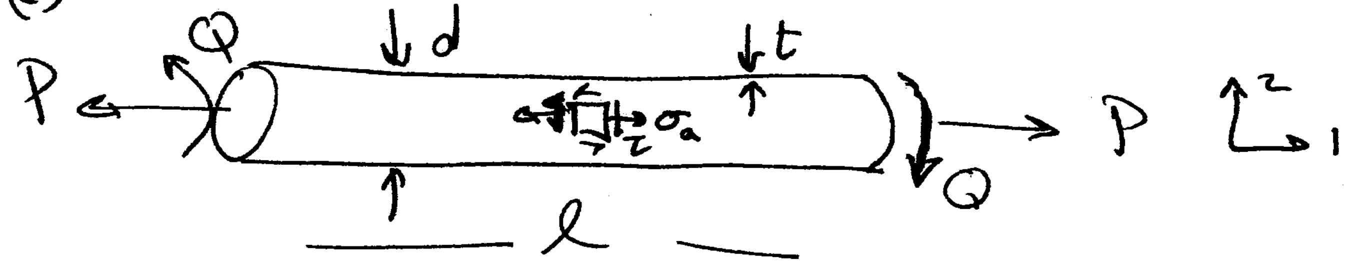
(b)

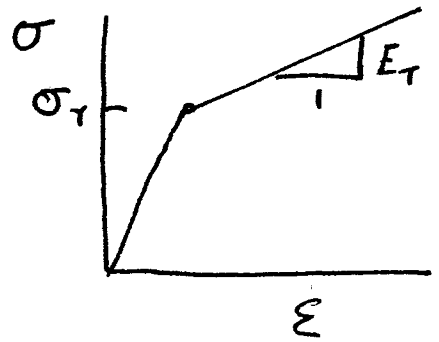


plastic deformation can occur between voids and give rise to a macroscopic change in volume due to the change in void size.

Consequently, the porous solid has a yield pressure and can undergo volumetric plastic straining.

(c)





$$\frac{1}{E_T} = \frac{d\varepsilon}{d\sigma} = \frac{1}{E} + \frac{1}{h}$$

$$\Rightarrow \quad E_T = \frac{Eh}{E+h}$$

(i)

$$\sigma_a \, \pi d t = P \quad \Rightarrow \quad \sigma_a = \frac{P}{\pi d t}$$

$$\tau \, \frac{\pi d^2 t}{2} = Q \quad \Rightarrow \quad \tau = \frac{2Q}{\pi d^2 t}$$

(ii) $\sigma_e^2 = \frac{3}{2} S_{ij} S_{ij}$ $\qquad \sigma_1 = \sigma_a \quad \sigma_2 = \sigma_3 = 0$

$S_{12} = \tau$

$$\Rightarrow \quad S_{11} = \frac{2}{3}\sigma_a \qquad S_{22} = S_{33} = -\frac{1}{3}\sigma_a$$

$$\Rightarrow \quad \sigma_e^2 = \frac{3}{2}\left(\frac{4}{9} + \frac{2}{9}\right)\sigma_a^2 + 3\tau^2 = \sigma_a^2 + 3\tau^2$$

$$\Rightarrow \quad \underline{\sigma_e = \left(\sigma_a^2 + 3\tau^2\right)^{1/2}}$$

Now $\quad Q = \alpha P d \quad \Rightarrow \quad \sigma_e^2 = \left(\frac{P}{\pi d t}\right)^2 + 3\left(\frac{2Q}{\pi d^2 t}\right)^2$

$$\Rightarrow \quad \sigma_e^2 = \left(\frac{P}{\pi d t}\right)^2 + 3\alpha^2\left(\frac{2Pd}{\pi d^2 t}\right)^2$$

(iii)

Case $\underline{\alpha = 0}$ $\Rightarrow$ $\sigma_e = \dfrac{P}{\pi d t}$

$\sigma_e = \sigma_Y \Rightarrow P_Y = \sigma_Y \pi d t$ $\quad \underline{\alpha = 0}$

Case $\underline{\alpha = 1}$: $\sigma_e = \sigma_Y \Rightarrow$

$$\sigma_Y^2 = \left(\frac{P}{\pi d t}\right)^2 + 12\left(\frac{P}{\pi d t}\right)^2 = 13\left(\frac{P}{\pi d t}\right)^2$$

$$\Rightarrow P = \frac{\sigma_Y \pi d t}{\sqrt{13}}$$

(iii) $\dot{\varepsilon}_{ij}^{PL} = \dfrac{3}{2} \dfrac{S_{ij}}{\sigma_e} \dfrac{\dot{\sigma}_e}{h}$

$\alpha = 1 \Rightarrow \sigma_e = \sqrt{13} \dfrac{P}{\pi d t} \Rightarrow \dot{\sigma}_e = \sqrt{13} \dfrac{\dot{P}}{\pi d t}$

$\Rightarrow \dot{\varepsilon}_{11}^{PL} = \dfrac{3}{2} \dfrac{S_{11}}{\sigma_e} \dfrac{\dot{\sigma}_e}{h}$

$\sigma_e = \sqrt{13}\left(\dfrac{P}{\pi d t}\right) \qquad S_{11} = \dfrac{2}{3}\sigma_a = \dfrac{2}{3}\dfrac{P}{\pi d t}$

$\Rightarrow \dot{\varepsilon}_{11}^{PL} = \dfrac{1}{\cancel{\sqrt{13}}} \cancel{\sqrt{13}} \dfrac{\dot{P}}{\pi d t h} \qquad \dot{\varepsilon}_{11}^{EL} = \dfrac{\dot{P}}{\pi d t E}$

$\Rightarrow \dot{\varepsilon}_{11} = \dot{\varepsilon}_{11}^{PL} + \dot{\varepsilon}_{11}^{EL} = \dfrac{\dot{P}}{\pi d t E_T} = \dfrac{\dot{u}}{l}$

$$\Rightarrow \frac{\dot{P}}{\dot{u}} = \frac{dP}{du} = \frac{\pi d t E_T}{l}$$

The least popular question but on the whole well answered. Students managed to employed the yield criterion correctly but found it hard to calculate the plastic hardening rates.