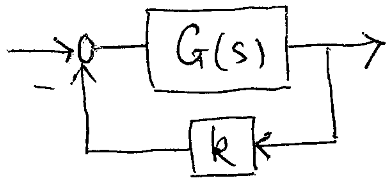
4F1 2011 Solutions - CONTROL SYSTEM DESIGN

1(a)(i) The root-locus diagram is the locus of roots of the feedback system below as k varies from 0 to ∞.



Roots satisfy: $G(s) + \frac{1}{k} = 0$ or $d(s) + k n(s) = 0$
where $G(s) = n(s)/d(s)$ are coprime polynomials.

(ii) Breakaway points are locations of coincident roots in the s-plane root-locus.

(iii) Given s_0 we find $k_0 = -1/G(s_0)$.

(b)(i) Taking Laplace transforms and rearranging:

$$\begin{cases} (m_s s^2 + cs + k)\hat{x} = (cs + k)\hat{y} \\ (m_u s^2 + cs + k + k_t)\hat{y} = (cs + k)\hat{x} \end{cases}$$

$$\Rightarrow (m_s s^2 + cs + k)(m_u s^2 + cs + k + k_t) - (cs + k)^2 = 0$$

which gives the required result after rearranging

(ii) Ch. eqn. takes the standard form for root-locus with $G(s) = n(s)/d(s)$ and

$$n(s) = s\left((m_s + m_u)s^2 + k_t\right)$$

$$d(s) = m_s m_u s^4 + (m_s k + m_s k_t + m_u k)s^2 + k k_t$$

(iii)

$$G(s) = \frac{9s(s^2+8)}{8(s^4+81s^2+72)}$$

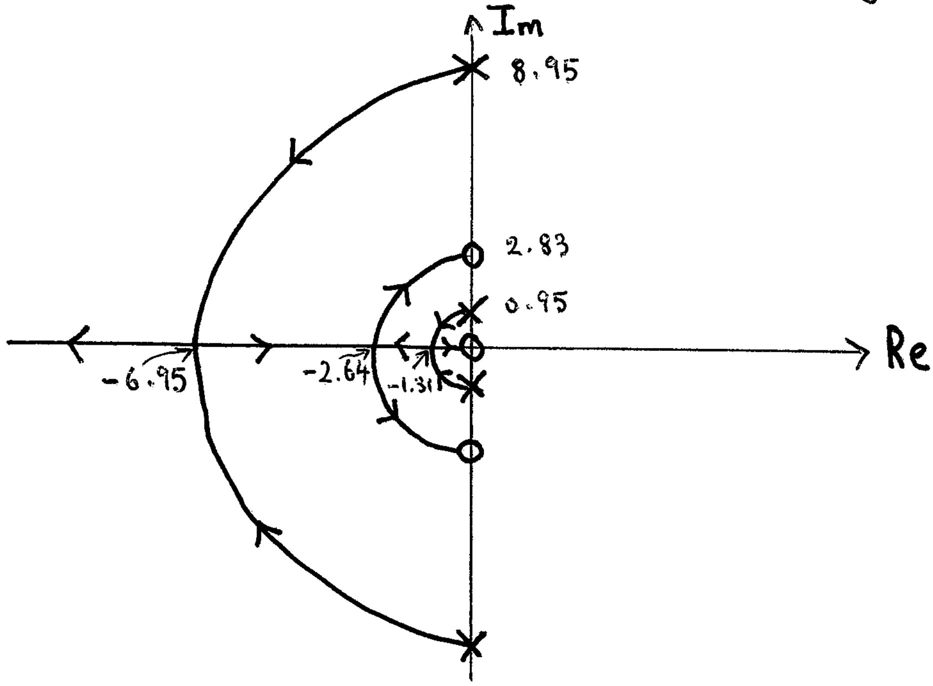
poles: $s^2 = -0.8989, -80.1011$

$\Rightarrow s = \pm j0.9481, \pm j8.9499$

zeros: $s = 0, \pm j2.8284$

Hence, zeros and poles alternate on imaginary axis

(iv)



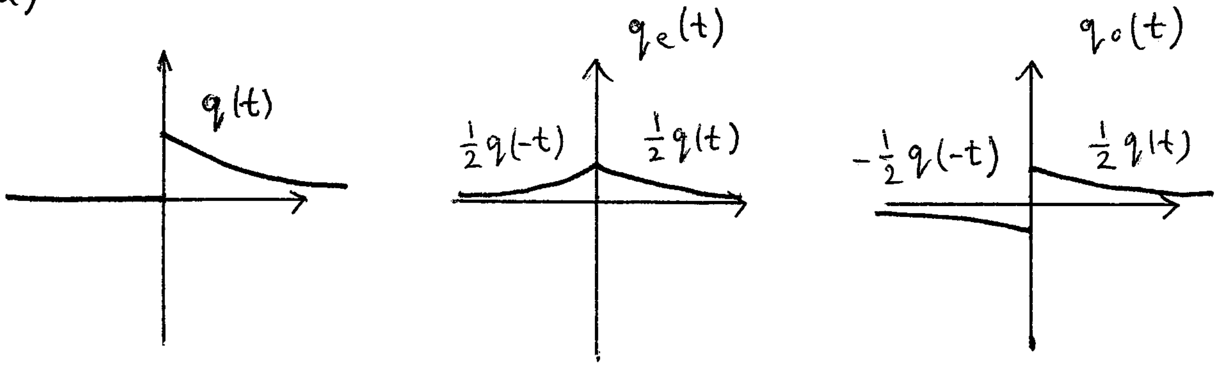
(v) $-1/G(-1.31) = 14.94$

$-1/G(-2.64) = 15.41$

$-1/G(-6.95) = 14.35$

All poles on negative real axis $\Leftrightarrow$ $14.94 < c < 15.41$

2 (a)



$Q(s) = \log G(s)$

$Q(j\omega) = \mathbb{F}(q(t))$

unique decomposition into $q_e(t)$ and $q_o(t)$

" transformation from " to "

$q_e(t)$ has purely real F.T. $Q_R(j\omega)$

$q_o(t)$ " " imaginary F.T. $Q_I(j\omega)$

$q_o(t) = \text{sign}(t)\, q_e(t)$

which implies a unique relationship between $Q_R(j\omega)$ & $Q_I(j\omega)$

and hence a unique relationship between $\log|G(j\omega)|$ & $\arg G(j\omega)$

(under appropriate conditions needed to ensure $\log G(s)$ is analytic in RHP)

(b) (i) See attached.

(b)(ii) All-pass factor has a lagging characteristic going from 0° to -180°. This suggests one zero in the RHP. Goes through -90° around $\omega = 5$ rad/sec suggesting a RHP zero around $s = 5$.

(c) (i)

To achieve SPEC B gain needs to be increased by about 6 dB at 0.1 rad s^{-1}.

To achieve SPEC C gain needs to be reduced by about 12dB at 10 rad s^{-1}.

2(c)(i) cont.

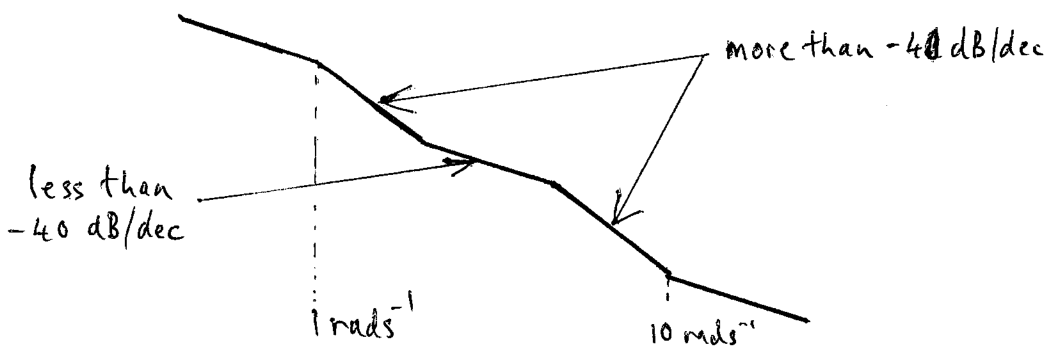
These gain changes could be achieved by a lag compensator, with pole and zero a decade (or more) apart acting in the frequency range between 0.1 and 10 rad s^{-1}.

To achieve SPEC A it is easiest if the crossover frequency is kept as low as possible within the range 0.1 to 10 rad s^{-1} where the plant has more phase in reserve.

Solution: place corner frequencies at 0.1 and 1 rad s^{-1} and then adjust gain. Choose:

$$K(s) = 0.3\left(\frac{s+1}{s+0.1}\right)$$

(c)(ii) SPECS B & C together require an average roll-off rate of −40 dB/dec between 1 and 10 rad s^{-1}. For constant slope this would incur a (minimum) phase penalty of −180°. In addition there is a significant additional phase lag ($\sim 55°$) around expected crossover from all-pass factor resulting from the RHP zero. A loop shape along the following lines would need to be tried:

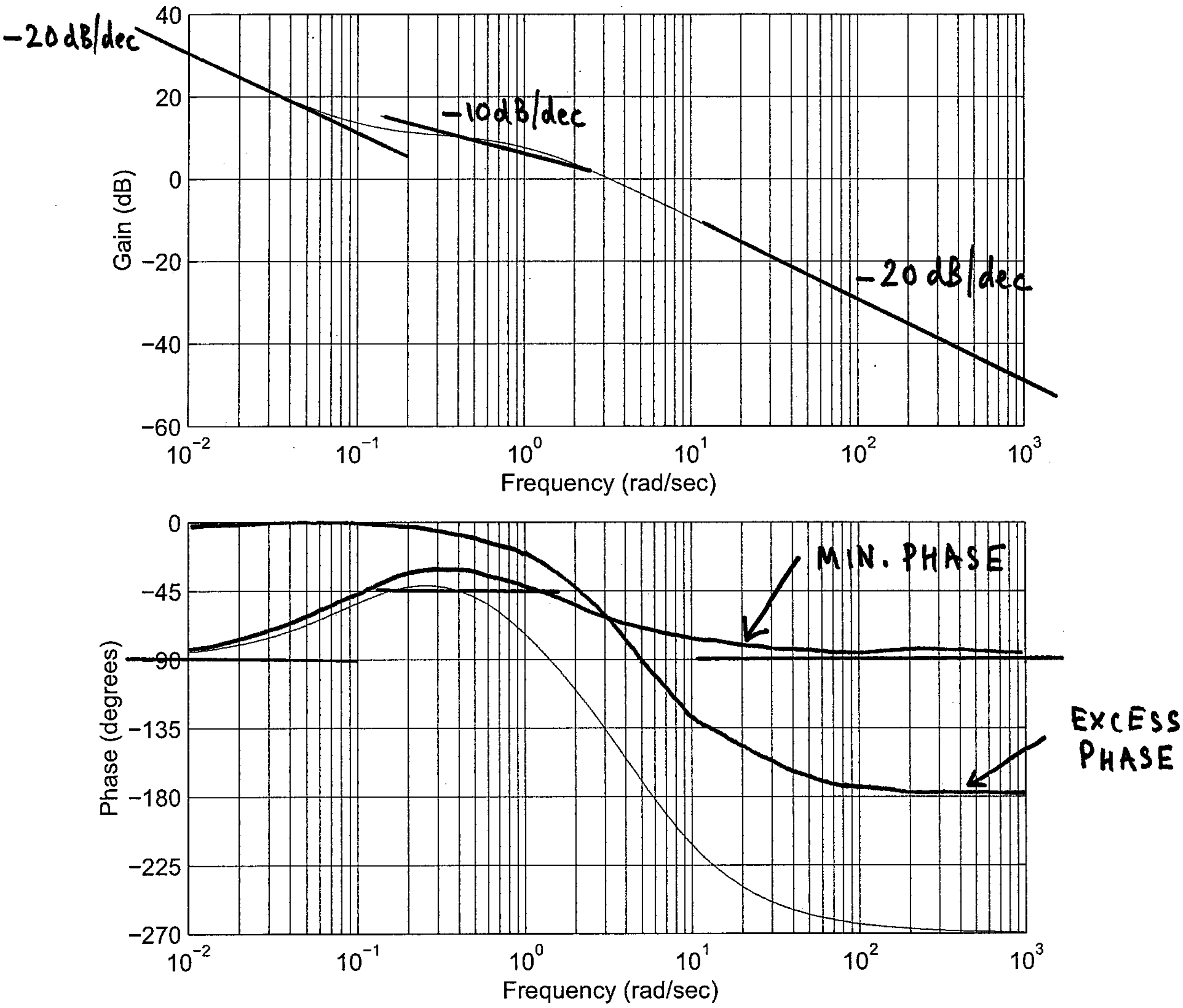


But it would be very difficult to achieve about +100° this way, to get a PM of 45°.

2(b)(i)

ENGINEERING TRIPOS PART IIB

~~Wednesday 27 April 2011~~, Module 4F1, Question 2.



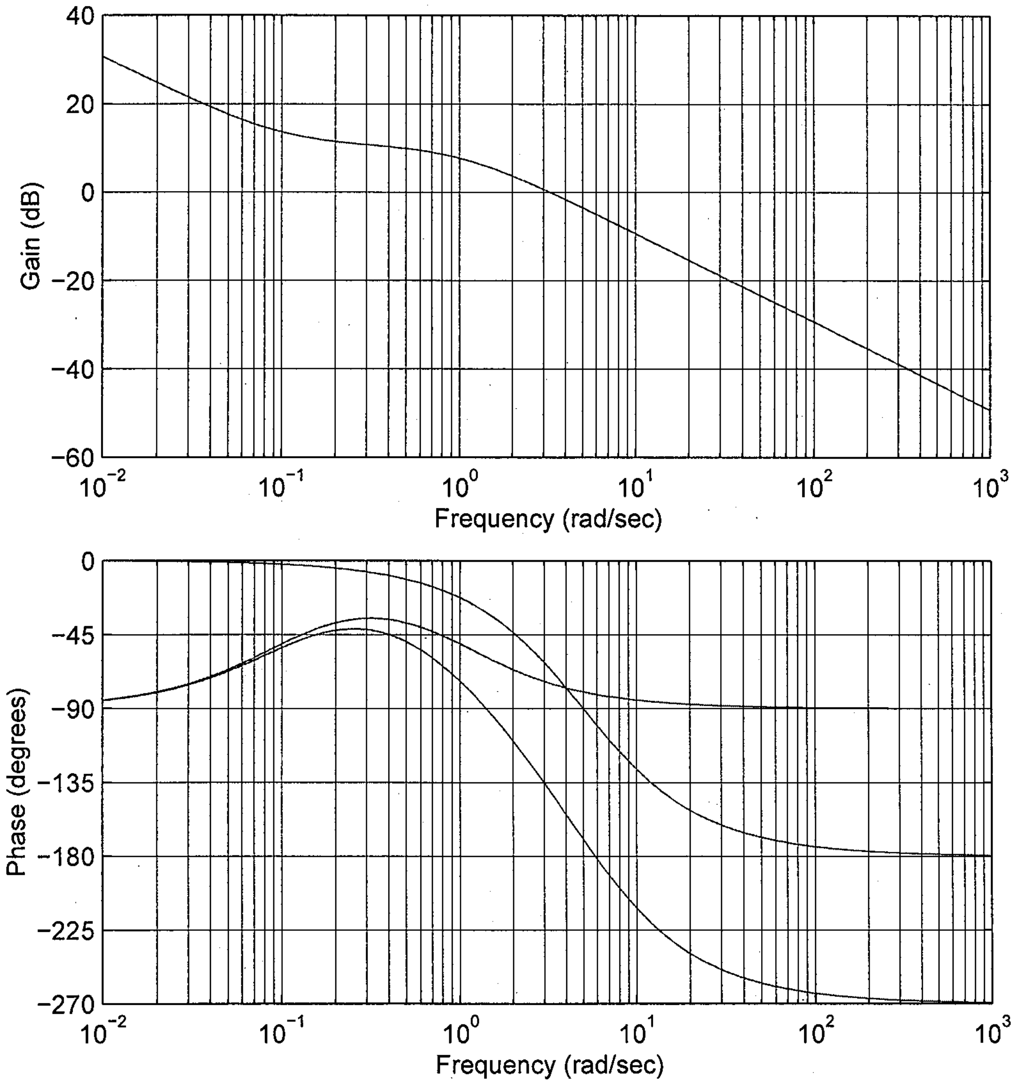
Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

mcs01

ACCURATE COMPUTER PLOT
2(b)(i)

ENGINEERING TRIPOS PART IIB

~~Wednesday 27 April 2011~~, Module 4F1, Question 2.



Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

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2(c)(i)

ENGINEERING TRIPOS PART IIB

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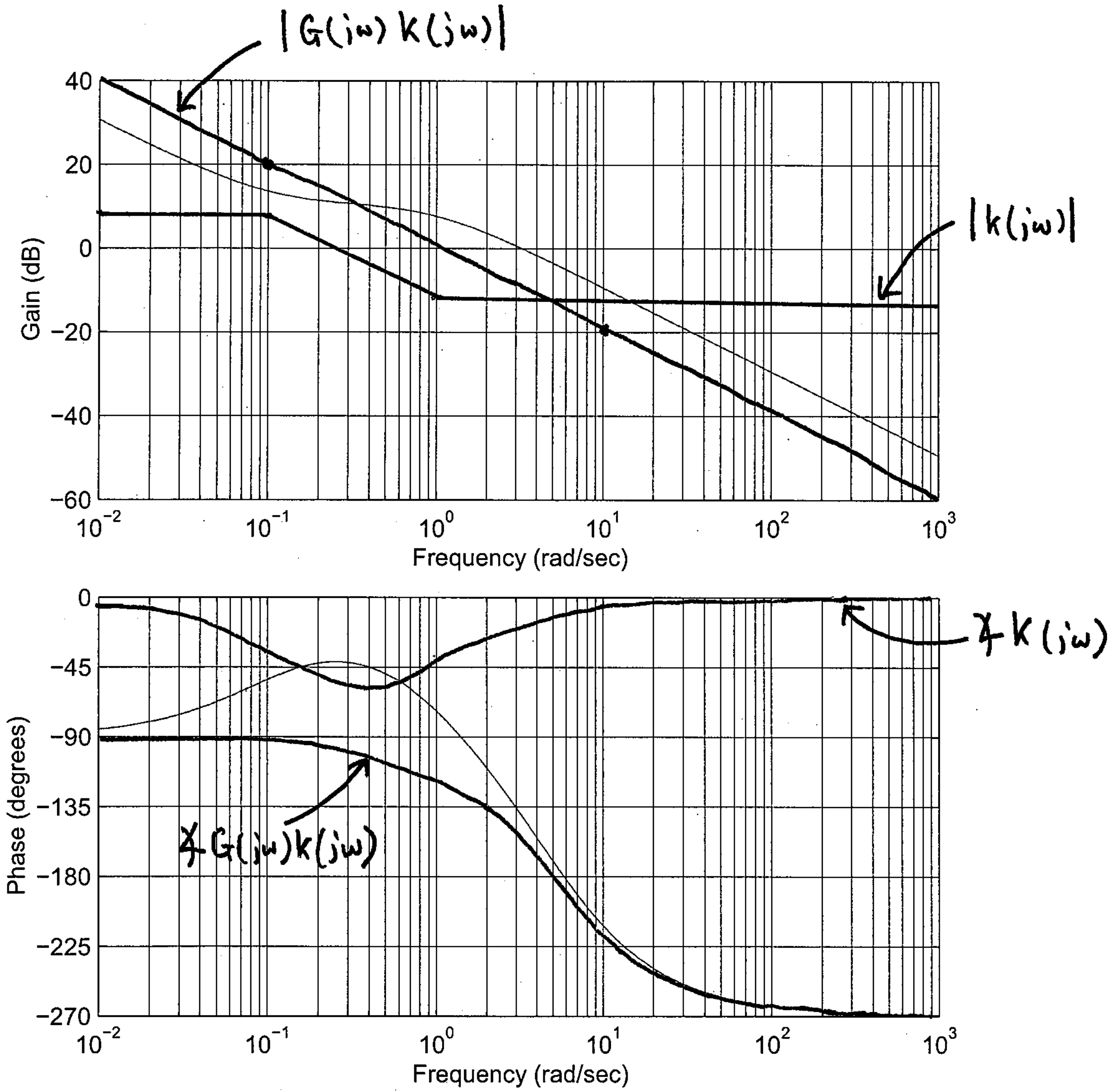

Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

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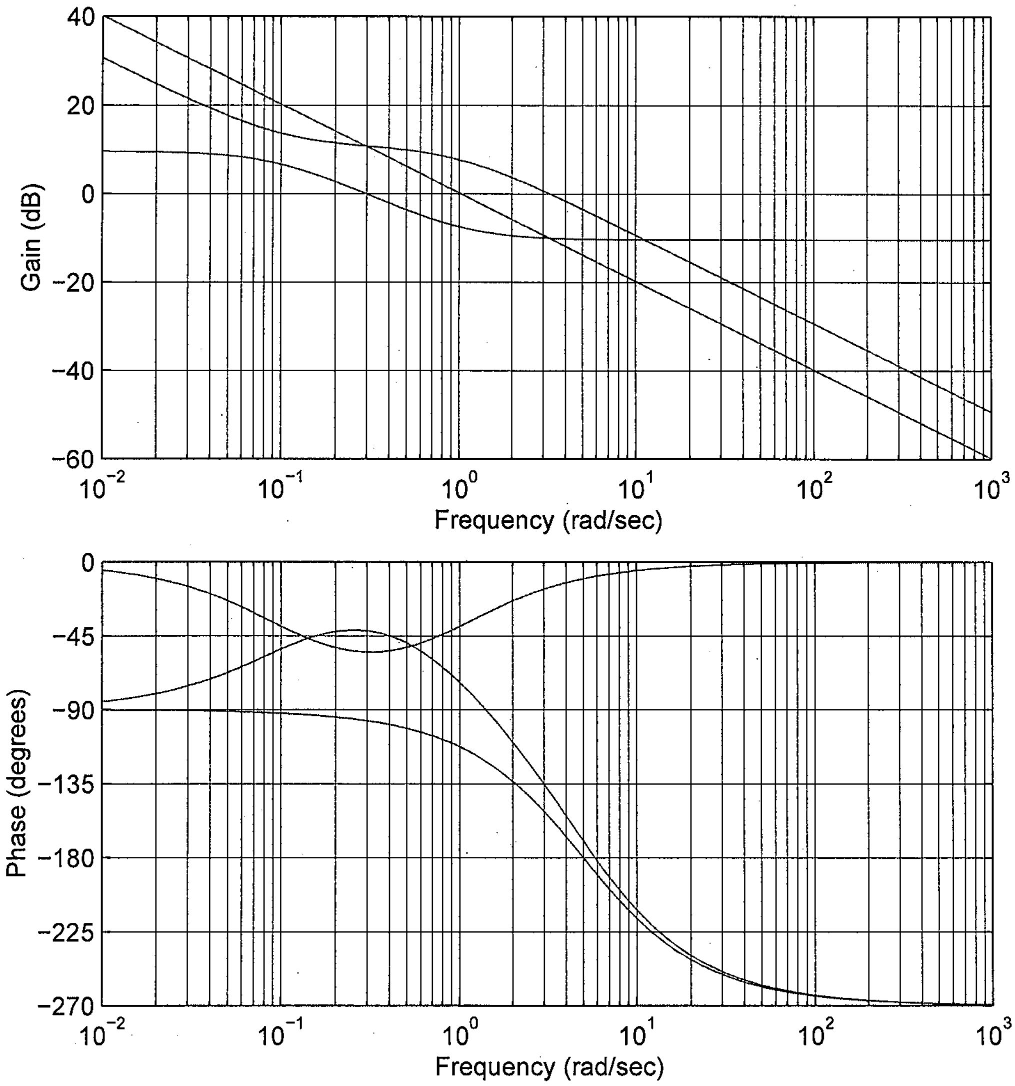
ACCURATE COMPUTER PLOT

2 (c) (i)

ENGINEERING TRIPOS PART IIB

~~Wednesday 27 April 2011~~, Module 4F1, Question 2.



Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

3. (a) RHP pole and zero means achievable stability margins will be poor. The sensitivity and disturbance rejection will also be poor.

Since the RHP pole lies to the right of the RHP zero the plant can only be stabilised with an unstable controller.

(b)
$$GK_1 = \frac{s-1}{(s-3)(s+2)} \frac{8(s+2)}{s-3} = \frac{8(s-1)}{(s-3)^2}$$

Closed-loop poles: $s^2 - 6s + 9 + 8(s-1) = 0$

$$\Leftrightarrow (s+1)^2 = 0 \quad (\text{and } s = -2)$$

(c)
$$GK = GK_1 \left(1 - \frac{cs}{s^2 + cs + 400}\right)$$

Multiplicative uncertainty model. Robust stability condition derived from Small Gain Theorem says closed loop will be stable providing

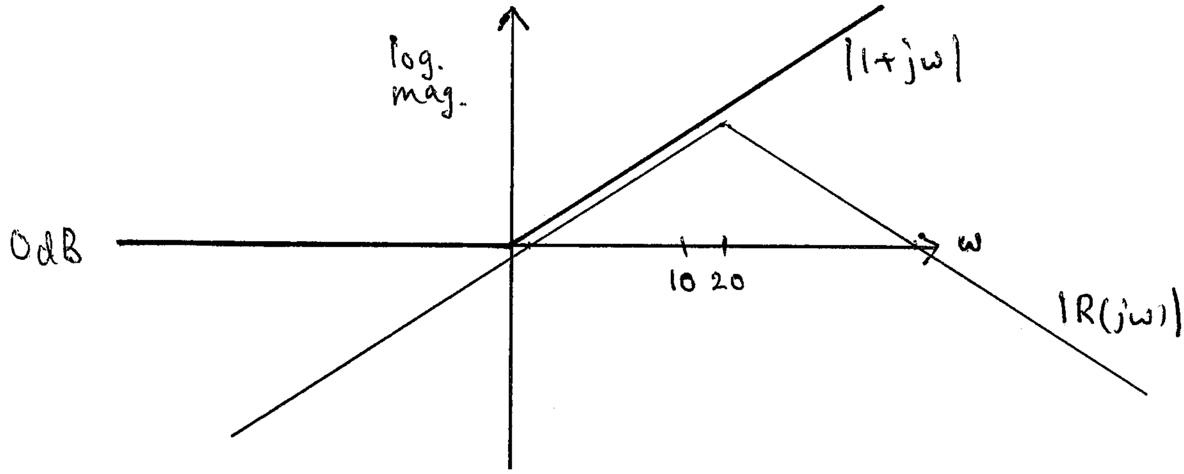
$$|\Delta(j\omega)| < \left| \frac{GK_1}{1 + GK_1} \right|^{-1}$$

$$\Leftrightarrow \left| \frac{cj\omega}{-\omega^2 + cj\omega + 400} \right| < \left| \frac{8(s-1)}{(s+1)^2} \right|^{-1}_{s=j\omega}$$

for all ω. This reduces to (1) since $\frac{s-1}{s+1}$ is all-pass.

(d) With $c = 40$ compare Bode magnitudes of

$$R(s) = \frac{320s}{(s+20)^2} \quad \text{and} \quad 1+s$$

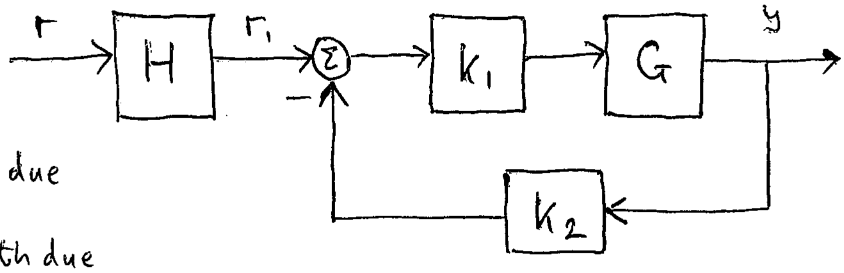


$$|R(j1)| = \frac{320}{1+20^2} < 1 \quad \text{and} \quad |R(j20)| = 8 < |1+20j|.$$

Hence inequality is satisfied for all ω.

$$\left(\text{Alter: } |R(j\omega)| = \frac{320\omega}{400+\omega^2} \leq \frac{320}{400}\omega \leq \omega \leq |1+j\omega|.\right)$$

(e)



k_1 in forward path due to RHP pole,
k_2 in feedback path due to imag. axis zeros.

$$T_{r_1 \to y} = \frac{Gk_1}{1+Gk_1k_2} = \frac{\dfrac{8(s-1)}{(s-3)^2}}{1 + \dfrac{8(s-1)}{(s-3)^2}\dfrac{s^2+400}{(s+20)^2}}$$

$$= \frac{8(s-1)(s+20)^2}{(s-3)^2(s+20)^2 + 8(s-1)(s^2+400)}$$

$$\Rightarrow H = -\frac{(s^4 + 42s^3 + 161s^2 + 1160s + 400)}{8(s+20)^2(5s+1)^2}$$

(LHP roots by design)

Principal Assessor's comments on 4F1

Question 1. Most candidates were successful with part (a) and part (b)(i)-(iii). A number of candidates were not able to deduce the correct form for the full root-locus diagram. A majority of candidates were able to get at least some way with the final part.

Question 2. This question was answered fairly well by most candidates. The main design task in Part (c)(i) was approached in an overly complex manner by many candidates who failed to notice that a lag compensator was sufficient to do the job. Few candidates noticed both the main points required in Part (c)(ii): the phase penalty from the steeper roll off between ω_1 and ω_2 and from the right half plane zero.

Question 3. Most candidates were able to do parts (a)-(c). Part (d) caused difficulties for some candidates who made elementary mistakes in sketching Bode diagrams. Few candidates successfully completed part (e). Disappointingly few noticed that K_1 should be placed in the forward path and K_2 in the feedback path.

M.C. Smith, 6 June 2011