PART IIB 2011

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4F2: ROBUST AND NONLINEAR SYSTEMS AND CONTROL

1. (a) For $\dot{x} = f(x)$:

i. An equilibrium point x_0 is a state such that $f(x_0) = 0$.

ii. If S is a set, such that $x(t) \in S \Rightarrow x(t+\tau) \in S$ for every $\tau > 0$ then S is an invariant set. (Also sometimes called a 'positively invariant set'.)

(b) LaSalle's Theorem: Let $S \subseteq \Re^n$ be a compact invariant set. Assume there exists a differentiable function $V : S \to \Re$ such that

$$\dot{V}(x) \leq 0 \; \forall x \in S$$

Let M be the largest invariant set contained in $\{x \in S \mid \dot{V}(x) = 0\}$ (the set of $x \in S$ for which $\dot{V}(x) = 0$). Then all trajectories starting in S approach M as $t \to \infty$.

Corollary: If the set $\{x \in S \mid \dot{V}(x) = 0\}$ contains no trajectories other than $x(t) = 0$, then 0 is locally asymptotically stable. Moreover, all trajectories starting in S converge to 0.

When using Lyapunov's theorems to establish asymptotic stability of an equilibrium $x = 0$, it is necessary to show that $\dot{V}(x) = 0$ at $x = 0$ only. In many practical examples this is not true but the conditions of LaSalle's theorem, which are weaker, *are* satisfied. LaSalle's Theorem can also be applied to establish convergence to more general invariant sets, such as limit cycles.

(c) i. At an equilibrium point $\dot{x}_1 = 0$, hence $x_2 = 0$ from the first state equation. Therefore at an equilibrium $\dot{x}_2 = -f(0) - g(x_1) = 0$, from the second state equation. But $f(0) = 0$, hence we must have $g(x_1) = 0$, and hence $x_1 = 0$ from the properties of g.

ii. The given function $V(x_1, x_2)$ satisfies the following properties:

$$\begin{aligned}
V(0,0) &= 0 \text{ by inspection,} && (1)\\
V(x_1,x_2) &> 0 \text{ for } (x_1,x_2) \neq (0,0), |x_1| \leq \sigma_0 \text{ and } |x_2| \leq \sigma_0, \text{ since} && (2)\\
& \quad \text{either } x_2^2 > 0 \text{ or } \int_0^{x_1} g(\sigma)d\sigma > 0, && (3)\\
& \quad \text{the latter following from the fact that } \sigma g(\sigma) > 0 \text{ for } \sigma \neq 0. && (4)\\
V(x_1,x_2) & \quad \text{is continuous in } x_1 \text{ and } x_2. && (5)
\end{aligned}$$

Hence $V(x_1, x_2)$ is a Lyapunov function candidate.
Along trajectories of the system we have

$$\dot{V} = \nabla V.f(x) = \begin{bmatrix} g(x_1) & x_2 \end{bmatrix} \begin{bmatrix} x_2 \\ -f(x_2) - g(x_1) \end{bmatrix} = -x_2 f(x_2) \leq 0 \qquad (6)$$

Hence $x = 0$ is a stable equilibrium point, by Lyapunov's Theorem.
However, $\dot{V} = 0$ whenever $x_2 = 0$, even if $x_1 \neq 0$, so asymptotic stability cannot be proved by Lyapunov's theorems.
Consider the set $S = \{(x_1, x_2) : V(x_1, x_2) \leq V_0, |x_1| \leq \sigma_0, |x_2| \leq \sigma_0\}$, where $V_0 > 0$ is some value. This set is invariant, since $\dot{V} \leq 0$. It is also compact (ie closed and bounded). So, by LaSalle's Theorem, all trajectories which start in or enter this set approach the largest invariant set within it for which $\dot{V} = 0$.
But $\dot{V} = 0 \Rightarrow x_2 = 0$; for this to remain true along a trajectory requires that both $x_1 = const$ (since $\dot{x}_1 = x_2$) and $\dot{x}_2 = 0$. As in part 1(c)i, this implies that $x_1 = 0$. Thus the only invariant set within S for which $\dot{V} = 0$ is the point $(x_1, x_2) = (0, 0)$, so by the corollary to LaSalle's Theorem this point is asymptotically stable.

Quite a tricky question on the stability of a mass on a nonlinear spring, with some subtleties. Popular and well answered, with the majority of students using LaSalle correctly.

2. (a) Suppose that, for a signal $e(t)$, we have $e(t) = E\sin(\omega t)$. If this is the input to the nonlinearity $f_1(e) = e^3$ the its output is $E^3\sin^3(\omega t)$. To find the describing function we need to find the fundamental (first harmonic) Fourier component of this.
Using the given formula, $\sin^3\theta = (3\sin\theta - \sin 3\theta)/4$.
Hence

$$E^3\sin^3(\omega t) = \frac{E^3}{4}[3\sin(\omega t) - \sin(3\omega t)] \tag{7}$$

so the fundamental component is $3E^3\sin(\omega t)/4$. Thus we have the describing function

$$N(E) = \frac{3E^3}{4E} = \frac{3E^2}{4} \tag{8}$$

(b) The describing function predicts limit cycles when the Nyquist plot intersects the plot of $-1/N(E)$. In this case the plot of $-1/N(E)$ lies on the negative real line, starting at $-\infty$ for $E = 0$, and increasing monotonically to 0 at $E = \infty$. It intersects the given Nyquist plot at the $\omega = 0$ point, and at the point where the Nyquist plot crosses the negative real line. (Not at the origin, because neither plot quite gets there for finite E and ω.) The $\omega = 0$ point does not indicate a limit cycle, since the frequency is 0.
Now we investigate the limit cycle indicated by the intersection at $\omega \neq 0$:
Frequency: This is the frequency at which $\operatorname{Im} G(j\omega) = 0$. Since the numerator of $G(j\omega)$ is real, it is enough to look at the denominator $d(j\omega)$:

$$d(j\omega) = (j\omega - 1)(j\omega + 5)^2 = (j\omega - 1)(-\omega^2 + 10j\omega + 25) \tag{9}$$

hence

$$\begin{aligned}
\operatorname{Im} d(j\omega) &= \omega(-\omega^2 + 25) - 10\omega && (10)\\
&= -\omega(\omega^2 - 15) && (11)\\
&= 0 \quad \text{for} \quad \omega = 0 \text{ and } \omega = \sqrt{15}, && (12)
\end{aligned}$$

so a limit cycle exists with frequency $\omega = \sqrt{15}$.
Amplitude: At the point of intersection we have

$$|G(j\sqrt{15})| = \frac{k}{\sqrt{15+1}(15+25)} = \frac{k}{4\times 40} = \frac{k}{160} \tag{13}$$

Hence at the intersection we have

$$-\frac{1}{N(E)} = -\frac{4}{3E^2} = -\frac{k}{160} \tag{14}$$

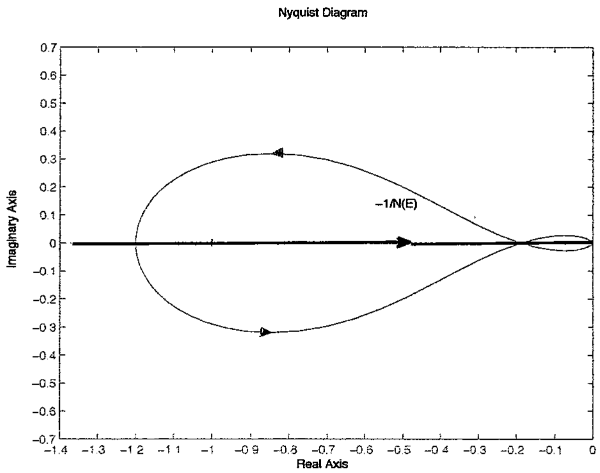

Figure 1: Plot of $-1/N(E)$ for Q.2b

and hence

$$E = \sqrt{\frac{4}{3} \times \frac{160}{k}} = 8\sqrt{\frac{10}{3k}} \tag{15}$$

Thus a limit cycle is predicted to exist with frequency $\sqrt{15}$ and amplitude (at the input to the nonlinearity) of $8\sqrt{10/3k}$.

However: This limit cycle is unstable, by the following argument. If the limit cycle amplitude E increases, then the point $-1/N(E)$ moves to the right of the point of intersection of the two plots. Considering this point as the "$-1/gain$" point of the Nyquist stability criterion predicts an unstable closed loop (because $G(s)$ is unstable), hence the amplitude may be expected to increase further. If the amplitude decreases slightly then a stable closed loop is predicted (the Nyquist locus encircles the point $-1/N(E)$ once counter- clockwise, if negative frequencies are included, and $G(s)$ has one pole) so the amplitude may be expected to decrease further.

Thus the limit cycle is very unlikely to be sustained for any length of time, and may not be observed at all.

(c) The input-output characteristic of the nonlinearity f_2 is shown in Fig.2. Note that the 'gain' $1/(1+|e|)+1$ *decreases* monotonically with E, from 2 (when $|e|=0$) to 1 (when $|e|=\infty$). Therefore $N(E)$

for this nonlinearity will decrease with E, also from 2 to 1 (considering the describing function as an 'equivalent gain'). Hence $-1/N(E)$ will decrease from $-1/2$ to -1 — that is, the plot evolves in the opposite direction to that in part (b). The relative directions of intersection of $G(j\omega)$ and $-1/N(E)$ will therefore also be opposite, and by the same argument as above, any predicted limit cycle will be stable.

Such a limit cycle will be predicted if $1/2 < |G(j\sqrt{15})| < 1$ (so that there is an intersection with the $-1/N(E)$ plot), namely if $1/2 < \frac{k}{160} < 1$, or $80 < k < 160$. However the amplitude of the limit cycle cannot be predicted without further calculation. The frequency remains at $\sqrt{15}$.

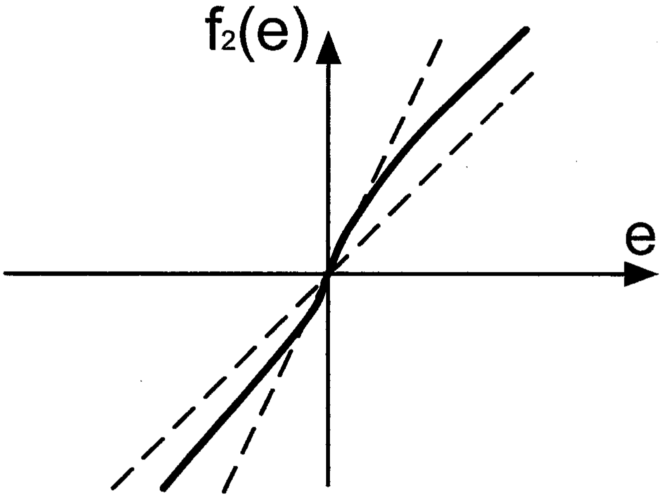

Figure 2: Input-output characteristic of $f_2(e)$ for Q.2c

(d) The Circle Criterion: Suppose the nonlinearity in Fig.2 of the question is a sector nonlinearity in the sector (α, β). Consider the circle which has the line segment $[-1/\beta, -1/\alpha]$ as diameter. The feedback system is asymptotically stable if the Nyquist locus of $G(j\omega)$ (plotted for $-\infty < \omega < +\infty$) encircles this circle as many times counter-clockwise as $G(s)$ has unstable poles.

In this case $G(s)$ has one unstable pole. Therefore one counter-clockwise encirclement is required to guarantee asymptotic stability.

In the case of $f_1(e)$ (part (b)), we have $\alpha = 0$ and $\beta = \infty$, so the whole negative real line would need to be encircled. Clearly this is impossible (cf Fig.1 of the question) for any value of k. So the circle criterion is not applicable to this problem.

In the case of $f_2(e)$ (part (c)), we have $\alpha = 1$ and $\beta = 2$. Thus the circle whose diameter is the segment $[-1, -1/2]$ of the real line would need to be encircled. This is possible if $k > 25$ (so that $G(0) < -1$) and $k < 80$ (so that $G(j\sqrt{15}) > -1/2$). The Nyquist plot shown in the question shows that the encirclement condition is satisfied for $k = 30$, so there is a range of values of k in the neighbourhood of 30 for which asymptotic stability of the closed loop can be deduced.

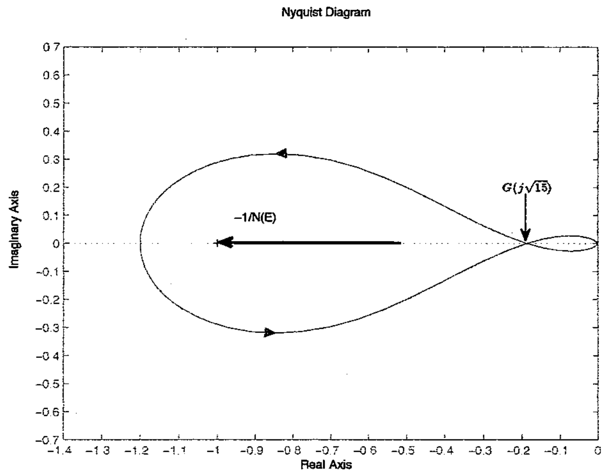

Figure 3: Plot of $-1/N(E)$ for Q.2c

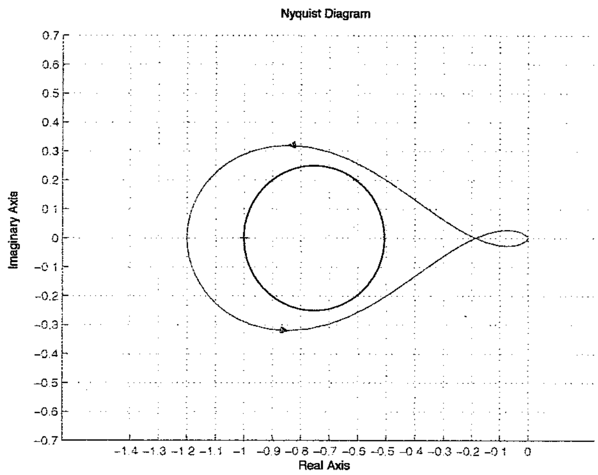

Figure 4: Encirclement of the circle with diameter $[-1, -1/2]$, for Q.2d.

Another popular question. The Fourier analysis to find the describing function in part a) was fine, but many students got lost in the details of b) and found the more general parts c) and d) too hard.

3 a) Bookwork

b) i) $\max\{\frac{1}{2}, \frac{1}{3}\} = \frac{1}{2}$

ii) $\left(\max_{\omega} \frac{1}{(1-\omega^2)^2 + 0.01\omega^2}\right)^{1/2}$

$= \left(1/\min_{\omega} (1-\omega^2)^2 + 0.01\omega^2\right)^{1/2} = 10.0125$ (at $\omega^2 = 0.9$

iii) Not in H_∞

iv) need $\bar{\sigma}\left(\begin{bmatrix}1&2\\2&1\end{bmatrix}\right) = \bar{\lambda}^{1/2}\left(\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1&2\\2&1\end{bmatrix}\right) = \bar{\lambda}^{1/2}\begin{bmatrix}5&4\\4&5\end{bmatrix}$

$= 3$

c) $\|P\|_\Delta = \|\sum_k \lambda_k P_k\|_\infty \le \lambda_1 \|P_1\| + \|\sum_{k>1} P_k\|_\Delta$ by triangle inequality

$\le \sum_k \lambda_k \|P_k\|_\Delta$ by repeated application of Δ inequality

$\le \max_k \|P_k\|_\Delta$

$\Rightarrow$ Need $\|C_\Delta\| < \frac{1}{\max_k \|P_k\|_\Delta}$

Not necessary, eg $P_1 = \frac{1}{s+1}$, $P_2 = \frac{1}{s+2}$

both stabilized by $c = -\frac{1}{s}$

Less popular question, but well answered - causing few problems.

4) a)

$$u = z + C(w + Pu)$$

$$\Rightarrow (I - CP)u = z + Cw$$

$$\Rightarrow u = (I - CP)^{-1} [C \quad I] \begin{bmatrix} w \\ z \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} y \\ u \end{bmatrix} = \underbrace{\begin{bmatrix} P \\ I \end{bmatrix} (I - CP)^{-1} [C \quad I]}_{H} \begin{bmatrix} w \\ z \end{bmatrix}$$

b) Need $\mu\left(\begin{bmatrix} \frac{1}{\gamma}I & \\ & \frac{1}{\epsilon}I \end{bmatrix} H(j\omega)\right) < 1 \quad \forall \omega$

(can be shown using the block diagram)

c) Again, from the block diagram,

$$P_s = P(I + \Delta)^{-1}, \quad \|\Delta\|_\Delta < \frac{1}{\gamma}$$

[this is quite tricky to spot]

Unpopular question. Straightforward if you understood what the question was asking, but most candidates didn't.