4F5 Advanced Wireless Communications, 2011 Crib

Question 1

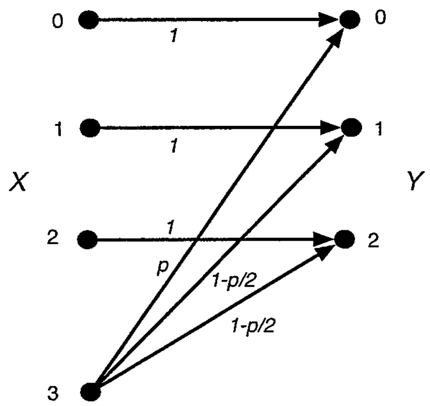


Figure 1: Channel model for Question 1 (a).

(a) *Consider the channel described in Fig. 1.*

(a) *Find the channel capacity of the channel.*
The capacity is $C = \log 3$, since the first 3 inputs are received without error.

(b) *How does the channel capacity depend on p?*
It does not depend on p.

(c) *What is the capacity-achieving input distribution?*
The optimal input distribution is $P_X(0) = P_X(1) = P_X(2) = \frac{1}{3}$.

(b) *Show that*

$$\Pr\{i(\bar{\boldsymbol{X}}, \boldsymbol{Y}) > \log_2 \beta\} \leq \frac{1}{\beta}$$

where

$$i(\boldsymbol{x}, \boldsymbol{y}) = \log_2 \frac{P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})}{P_{\boldsymbol{Y}}(\boldsymbol{y})}$$

and $\bar{\boldsymbol{X}}$ *and* $\boldsymbol{Y}$ *are independent.*

We have that

$$\begin{aligned}
\Pr\{i(\bar{\boldsymbol{X}}, \boldsymbol{Y}) > \log_2 \beta\} &= \Pr\left\{\frac{P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{Y}|\bar{\boldsymbol{X}})}{P_{\boldsymbol{Y}}(\boldsymbol{Y})} > \beta\right\} && (1)\\
&= \sum_{\boldsymbol{x},\boldsymbol{y}: P_{\boldsymbol{Y}}(\boldsymbol{y}) < \frac{1}{\beta} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})} P_{\boldsymbol{X}}(\boldsymbol{x}) P_{\boldsymbol{Y}}(\boldsymbol{y}) && (2)\\
&\leq \sum_{\boldsymbol{x},\boldsymbol{y}: P_{\boldsymbol{Y}}(\boldsymbol{y}) < \frac{1}{\beta} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x})} P_{\boldsymbol{X}}(\boldsymbol{x}) \frac{1}{\beta} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) && (3)\\
&\leq \sum_{\boldsymbol{x},\boldsymbol{y}} P_{\boldsymbol{X}}(\boldsymbol{x}) \frac{1}{\beta} P_{\boldsymbol{Y}|\boldsymbol{X}}(\boldsymbol{y}|\boldsymbol{x}) && (4)\\
&= \frac{1}{\beta} \sum_{\boldsymbol{y}} P_{\boldsymbol{Y}}(\boldsymbol{y}) && (5)\\
&= \frac{1}{\beta} && (6)
\end{aligned}$$

(c) *Hence show that the average error probability of a code with M codewords can be upper-bounded as*

$$\bar{P}_e \leq \Pr\{i(\boldsymbol{X}, \boldsymbol{Y}) \leq \log_2 \beta\} + \frac{M-1}{\beta}.$$

Following the lectures, we employ a threshold decoder for which the decoder decides for message m if m is the only message for which $i(\boldsymbol{x}_m, \boldsymbol{y}) > \log_2 \beta$; it declares an error otherwise. Then, with random coding we have that

$$P_e(m) = \Pr\left\{ \bigcup_{\ell \neq m} \big(i(\boldsymbol{X}_\ell;\boldsymbol{Y}) > \log_2 \beta\big) \cup \big(i(\boldsymbol{X}_m;\boldsymbol{Y}) \leq \log_2 \beta\big) \middle| \, \mathsf{m} = m \right\} \tag{7}$$

$$\leq \sum_{\ell \neq m} \Pr\big\{i(\boldsymbol{X}_\ell;\boldsymbol{Y}) > \log_2 \beta \mid \mathsf{m} = m\big\} + \Pr\big\{i(\boldsymbol{X}_m;\boldsymbol{Y}) \leq \log_2 \beta \mid \mathsf{m} = m\big\} \tag{8}$$

$$= (M-1)\Pr\big\{i(\bar{\boldsymbol{X}};\boldsymbol{Y}) > \log_2 \beta\big\} + \Pr\big\{i(\boldsymbol{X};\boldsymbol{Y}) \leq \log_2 \beta\big\} \tag{9}$$

$$\leq \Pr\{i(\boldsymbol{X},\boldsymbol{Y}) \leq \log_2 \beta\} + \frac{M-1}{\beta} \tag{10}$$

(d) *Using this bound, show the achievability part of Shannon's capacity theorem.*

We choose β such that $\Pr\big(i(\boldsymbol{X};\boldsymbol{Y}) \leq \log_2 \beta\big) \longrightarrow 0$ as $n \longrightarrow \infty$:

$$\log_2 \beta = n\big(I(X;Y) - \delta\big), \quad \text{for some arbitrary } \delta > 0$$

Indeed, with this choice we obtain

$$\begin{aligned} \Pr\big(i(\boldsymbol{X};\boldsymbol{Y}) \leq \log_2 \beta\big) &= \Pr\left(\frac{1}{n} i(\boldsymbol{X};\boldsymbol{Y}) - I(X;Y) \leq -\delta\right) \\ &\leq \Pr\left(\left|\frac{1}{n} i(\boldsymbol{X};\boldsymbol{Y}) - I(X;Y)\right| \geq \delta\right) \\ &\longrightarrow 0, \quad \text{as } n \longrightarrow \infty \end{aligned}$$

since $i(\boldsymbol{X};\boldsymbol{Y}) = \sum_{i=1}^{n} i(X_i;Y_i)$ and $\mathbb{E}[i(X;Y)] = I(X;Y)$.

The question proved very difficult for most students. Some students did not see that 3 channel input symbols were not affected by noise, while the 4th was always received in error. The mathematical derivations asked were all in the notes.

Question 2

Pulse-position modulation (PPM) is a modulation scheme that maps the information bits onto the position of a pulse within the signalling period T_s. In particular, M-PPM uses the following signals for transmission

$$x_m(t) = \begin{cases} 1 & (m-1)\frac{T_s}{M} \leq t \leq m\frac{T_s}{M} \\ 0 & \textit{otherwise.} \end{cases}$$

(a) *What is the dimension of the signal space?*

The dimension of the signal space is the number of linearly independent signals in the space. In this case, the dimension is M.

(b) *Find a suitable orthonormal basis for the signal space.*

The signal set is orthogonal, and therefore, an orthonormal basis is given by

$$f_m(t) = \begin{cases} \sqrt{\frac{M}{T_s}} & (m-1)\frac{T_s}{M} \leq t \leq m\frac{T_s}{M} \\ 0 & \text{otherwise.} \end{cases}$$

(c) *Write down the vector representation as a function of the basis.*

The signal vectors can be written as

$$\boldsymbol{x}_1 = \left(\sqrt{\frac{M}{T_s}}, 0, \ldots, 0\right), \boldsymbol{x}_2 = \left(0, \sqrt{\frac{M}{T_s}}, 0, \ldots, 0\right), \ldots$$

(d) *Find the energy of each signal vector and the corresponding average energy.*

The energy of each signal is given by $\frac{T_s}{M}$, and hence the average energy is the same.

(e) *Draw the block diagram of the optimal receiver*

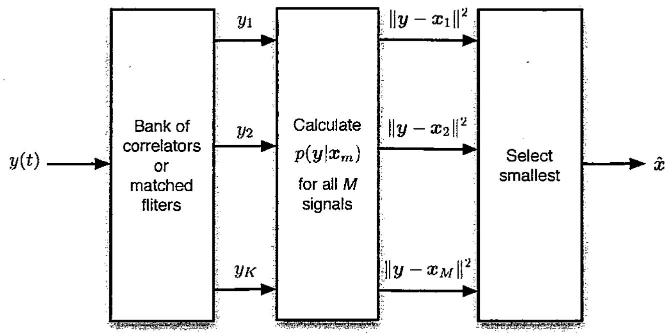
(f) *Write down the equations of the optimal detector and use the union bound to find an upper bound to the probability of error.*

Let $p(\boldsymbol{x}_m|\boldsymbol{y}) \triangleq \Pr\{\text{signal } \boldsymbol{x}_m \text{ was transmitted} \mid \boldsymbol{y} \text{ was received}\}$. The optimum detector selects the $\boldsymbol{x}_m$ that maximizes the a posteriori probability (MAP)

$$\hat{\boldsymbol{x}} = \arg \max_{m=1,\ldots,M} p(\boldsymbol{x}_m|\boldsymbol{y}) \tag{11}$$

$$= \arg \max_{m=1,\ldots,M} \frac{p(\boldsymbol{y}|\boldsymbol{x}_m)p(\boldsymbol{x}_m)}{p(\boldsymbol{y})} \tag{12}$$

$$= \arg \max_{m=1,\ldots,M} p(\boldsymbol{y}|\boldsymbol{x}_m)p(\boldsymbol{x}_m) \tag{13}$$



If all signals are equally likely, i.e., $p(\boldsymbol{x}_m) = 1/M$, for $m = 1, \ldots, M$, then

$$\hat{\boldsymbol{x}} = \arg \max_{m=1,\ldots,M} p(\boldsymbol{y}|\boldsymbol{x}_m) \tag{14}$$

$$= \arg \max_{m=1,\ldots,M} \log p(\boldsymbol{y}|\boldsymbol{x}_m) \tag{15}$$

$$= \arg \min_{m=1,\ldots,M} \|\boldsymbol{y} - \boldsymbol{x}_m\|^2 \tag{16}$$

To find a bound, we use the union bound

$$P_e = \frac{1}{M} \sum_{m=1}^{M} \Pr\{\text{error}|\boldsymbol{x}_m \text{ was transmitted}\} \tag{17}$$

$$= \frac{1}{M} \sum_{m=1}^{M} \Pr\left\{ \bigcup_{m' \neq m} \{\hat{\boldsymbol{x}} = \boldsymbol{x}_{m'}\} \middle| \boldsymbol{x}_m \right\} \tag{18}$$

$$\leq \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} \Pr\{\hat{\boldsymbol{x}} = \boldsymbol{x}_{m'}|\boldsymbol{x}_m\} \tag{19}$$

$$= \frac{1}{M} \sum_{m=1}^{M} \sum_{m' \neq m} Q\left(\sqrt{\frac{\|\boldsymbol{x}_m - \boldsymbol{x}_{m'}\|^2}{2N_0}}\right) \tag{20}$$

(g) *Explain why the union bound is not a probability, and explain why it is still an interesting bound to characterise the error probability.*

The error events are not disjoint. There are certain probabilities that are counted multiple times and this implies that the union bound may be greater than 1. However, at high SNR, the dominating component is the probability of error with the nearest neighbour (term at minimum Euclidean distance), since all other terms vanish due to the properties of the Q function. Therefore, at high SNR, the union bound provides an accurate estimate of the error probability.

Popular question, generally well answered.

Question 3

Consider the code whose generator matrix is given by

$$\boldsymbol{G} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

(a) *Write down the corresponding parity-check matrix in systematic form and the parity-check equations. What is the code rate?*

The code rate is $R = 3/7$. The parity check matrix in systematic form is given by

$$\boldsymbol{H} = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the parity-check equations are

$$x_1 + x_2 + x_4 = 0 \tag{21}$$
$$x_2 + x_3 + x_5 = 0 \tag{22}$$
$$x_1 + x_2 + x_3 + x_6 = 0 \tag{23}$$
$$x_1 + x_3 + x_7 = 0 \tag{24}$$

(b) *Write down the variable and check-node degree distribution polynomials (node perspective) of the code, interpreted as a low-density parity-check code, i.e., $\Lambda(x)$ and $P(x)$.*

By inspecting the parity-check matrix, we obtain that

$$\Lambda(x) = 4x + 3x^3 \tag{25}$$
$$P(x) = 3x^3 + x^4 \tag{26}$$

(c) *Find the minimum distance of the code.*

We enumerate all codewords and find the one at minimum distance. By inspection $d_{\min} = 4$.

(d) *The code is used to transmit information over a binary symmetric channel (BSC) with crossover probability $\varepsilon < 0.5$.*

Table 1: default

Input	Codeword
000	0000000
001	0010111
010	0101110
011	0111001
100	1001011
101	1011100
110	1100101
111	1110010

(a) *Show that finding the maximum likelihood (ML) codeword is equivalent to finding the codeword at minimum Hamming distance.*

ML decoding solves the following problem

$$\begin{aligned}
\hat{\boldsymbol{x}} &= \arg\max P_{\mathbf{Y}|\mathbf{X}}(\boldsymbol{y}|\boldsymbol{x}) && (27)\\
&= \arg\max P_{\mathbf{Y}|\mathbf{X}}(\boldsymbol{y}|\boldsymbol{x}) && (28)\\
&= \arg\max \prod_{i=1}^{n} P_{Y|X}(y_i|x_i) && (29)\\
&= \arg\max \varepsilon^{w}(1-\varepsilon)^{n-w} && (30)\\
&= \arg\max \log \varepsilon^{w}(1-\varepsilon)^{n-w} && (31)\\
&= \arg\max w\log\varepsilon + (n-w)\log(1-\varepsilon) && (32)\\
&= \arg\max n\log(1-\varepsilon) + w\log\varepsilon - w\log(1-\varepsilon) && (33)\\
&= \arg\max n\log(1-\varepsilon) - w\log\frac{1-\varepsilon}{\varepsilon} && (34)\\
&= \arg\min w && (35)
\end{aligned}$$

where w is the number of differences between a candidate $\boldsymbol{x}$ and the received word $\boldsymbol{y}$ (Hamming weight of the pairwise difference), and the last step follows from the fact that the terms $n\log(1-\varepsilon)$ and $\log\frac{1-\varepsilon}{\varepsilon} > 0$ are common to all codewords. Therefore, the ML decoder, finds the codeword that is closest in Hamming distance to the received word. Note that it is independent of ε.

(b) *The code is used to transmit 9 independent bits of information. Find the ML estimate if the received word is*

$$\boldsymbol{y} = [1\ 0\ 1\ 1\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 0\ 1\ 1\ 1\ 1\ 1\ 0\ 1\ 0].$$

The received sequence consists of three independent strings of length 7

$$\boldsymbol{y} = [\boldsymbol{y}_1 \ \boldsymbol{y}_2 \ \boldsymbol{y}_3]$$

since the input bits are independent and equiprobable and since the channel is memoryless. Also the encoder is memoryless, unlike the encoder of a convolutional encoder. Hence, the maximum likelihood sequence estimate is the maximum likelihood estimates of the three received strings. If we call the three transmitted codewords

$$\boldsymbol{c} = [\boldsymbol{c}_1 \ \boldsymbol{c}_2 \ \boldsymbol{c}_3]$$

the ML estimates of $[\boldsymbol{c}_1 \ \boldsymbol{c}_2 \ \boldsymbol{c}_3]$ are the codewords with minimum Hamming distances from $[\boldsymbol{y}_1 \ \boldsymbol{y}_2 \ \boldsymbol{y}_3]$. The estimated codewords and corresponding information bit strings are obtained from the table above. Therefore

$$\hat{\boldsymbol{c}}_1 = [1011100] \tag{36}$$

$$\hat{\boldsymbol{c}}_2 = [1100101] \tag{37}$$

$$\hat{\boldsymbol{c}}_3 = [1110010] \tag{38}$$

Which implies that the information bits are given by

$$\boldsymbol{b} = [1\ 0\ 1 \quad 1\ 1\ 0 \quad 1\ 1\ 1].$$

Popular question, very well answered.

Question 4

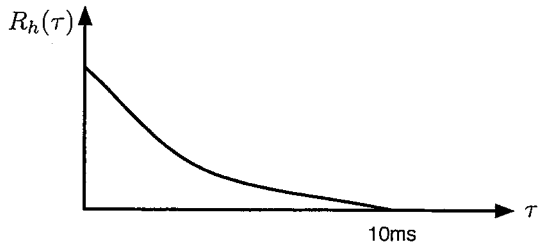


Figure 2: Multipath intensity profile for Question 4.

(a) *Consider transmission over a fading channel with the multipath intensity profile given in Figure 2 and Doppler spectrum following the Jakes model given by*

$$S_H(\xi) = \begin{cases} \frac{1}{\pi f_m} \frac{1}{\sqrt{1-(\xi/f_m)^2}} & |\xi| \leq f_m \\ 0 & |\xi| > f_m \end{cases}$$

where f_m is the maximum Doppler frequency.

(i) *What is the coherence bandwidth of the channel?*
The delay spread of the channel is $T_d = 10ms$. Therefore, the coherence bandwidth is $B_c = 1/T_d = 100Hz$.

(ii) *What is the coherence time of the channel if the carrier frequency is $f_c = 10MHz$ and the mobile user is moving at a velocity $v = 10km/h$?*
The maximum Doppler frequency is ($v = 2.77m/s$).

$$f_m = vf_c/c = 0.0925Hz$$

and therefore $T_c = 1/f_m = 10.8s$.

(iii) *Will the channel introduce frequency or time selectivity if codewords of duration $T_x = 20ms$ using signals of bandwidth $B_x = 2$ MHz are employed for transmission.*
Since $B_x \gg B_c$ the channel is frequency selective. Since $T_x \ll T_c$ the channel is not time selective.

(b) *Consider a multiple-input single-output (MISO) channel described by*

$$y = \boldsymbol{h}^T \boldsymbol{x} + z$$

where $\boldsymbol{h} = (h_1, \ldots, h_{n_t}) \in \mathbb{C}^{n_t}$ is the vector of fading coefficients, with n_t being the number of transmit antennas. Assume that the entries of $\boldsymbol{h}$ are zero mean, unit variance complex Gaussian random variables.

(i) *Suppose that at every time step, one symbol is transmitted, each time from a different antenna. What is the diversity order of this scheme? What is the rate?*

This is equivalent to a single-input single-output channel (SISO), since there is no multiple-antenna transmission as such. Therefore, the diversity order is 1. The rate is 1 symbol per channel use.

(ii) *Suppose that a repetition code is employed, and assume that the same symbol is transmitted out of each antenna during the same signalling period. What is the diversity order of this scheme? What is the rate?*

In this case, the transmitted vector is $\boldsymbol{x} = (x, \ldots, x)^T$, and therefore the channel model becomes

$$y = \boldsymbol{h}^T \boldsymbol{x} + z \tag{39}$$

$$= \boldsymbol{h}^T \begin{pmatrix} x \\ \vdots \\ x \end{pmatrix} + z \tag{40}$$

$$= \sum_{i=1}^{n_t} h_i \; x + z \tag{41}$$

In order to have a deep fade, each of the h_i needs to be in a deep fade. This implies a diversity order equal to n_t. The rate of the scheme is 1 symbol per channel use.

(iii) *Suppose now that $n_t = 2$ and that Alamouti's scheme is employed. Therefore, we have that*

$$[y_1 \; y_2] = [h_1 \; h_2] \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} + [n_1 \; n_2].$$

What is the diversity order achieved by this scheme? What is the rate?

Compute

$$\widetilde{y}_1 = h_1^* y_1 + h_2 y_2 = \left(|h_1|^2 + |h_2|^2\right) x_1 + \widetilde{n}_1 \tag{42}$$

$$\widetilde{y}_2 = h_2^* y_1 + h_1 y_2^* = \left(|h_1|^2 + |h_2|^2\right) x_2 + \widetilde{n}_2 \tag{43}$$

where $\tilde{n}_1, \tilde{n}_2$ are i.i.d. with distribution $\sim \mathcal{N}_{\mathbb{C}}(0, (|h_1|^2 + |h_2|^2)\sigma^2)$. Again, in order for the 2 subchannels defined by the previous equations to be in a deep fade, both h_1 and h_2 should be in a deep fade simultaneously. Hence the diversity order is 2. The rate is 2 symbols per 2 channel uses, which equals 1 symbol per channel use.

Rather simple question. The first part was straightforward, while the second required some more thinking about diversity/rate schemes.