Part IIA/IIB 4M12: PDEs and Variational Methods
2011 Crib 2011 DR GN WELLS
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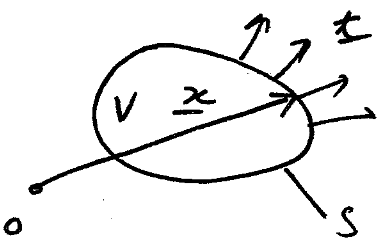
Q1 a) $\int_{\partial V} \underline{u}\cdot\underline{\underline{A}}\,\underline{n}\, dS = \int_{\partial V} u_i A_{ij} n_j\, dS$

$$= \int_V (u_i A_{ij})_{,j}\, dV \quad \text{(divergence theorem)}$$

$$= \int_V (A^T_{ji} u_i)_{,j}\, dV$$

$$= \int \nabla\cdot(\underline{\underline{A}}^T \underline{u})\, dV$$

b)



Moment equilibrium:

$$\int_{\partial V} \underline{x} \times \underline{t}\, dS + \int_V \underline{x} \times \rho \underline{g}\, dV = \underline{0}$$

→ insert $\underline{t} = \underline{\underline{\sigma}}\,\underline{n}$,

$$\int_{\partial V} \underline{x} \times \underline{\underline{\sigma}}\,\underline{n}\, dS + \int_V \underline{x} \times \rho \underline{g}\, dV = \underline{0}$$

→ use index notation,

$$\int_{\partial V} \varepsilon_{ijk} x_j \sigma_{kl} n_l\, dS + \int_V \varepsilon_{ijk} x_j \rho g_k\, dV = 0$$

Apply divergence theorem,

$$\int_V (\varepsilon_{ijk} x_j \sigma_{kl})_{,l} + \varepsilon_{ijk} x_j \rho g_k\, dV = 0$$

Expand, $\int_V \varepsilon_{ijk} x_{j,l} \sigma_{kl} + \varepsilon_{ijk} x_j \sigma_{kl,l} + \varepsilon_{ijk} x_j \rho g_k\, dV = 0$

$$\Rightarrow \int_V \varepsilon_{ijk} \delta_{jl} \sigma_{kl} + \varepsilon_{ijk} x_j \underbrace{(\sigma_{kl,l} + \rho g_k)}_{(=0 \text{ by force balance})} dV = 0$$

$= 0$ if $\sigma_{kl} = \sigma_{lk}$ by symmetry / anti-symmetry.

Q1 c) $J = \int_V (\nabla^2 w)^2 \, dV$

Take directional derivative and set equal to zero to find minimum,

$$DJ_v(w)[v] = \int_V 2(\nabla^2 w)\nabla^2 v \, dV = 0 \quad \forall v$$

Integrate by parts,

$$-2\int_V \nabla(\nabla^2 w)\cdot\nabla v \, dV + 2\int_{\partial V} \nabla v \cdot \underline{n}\, (\nabla^2 w)\, ds = 0$$

Integrate by parts again,

$$2\int_V \underline{\nabla\cdot\nabla(\nabla^2 w)}\, v \, dV + 2\int_{\partial V} \nabla v \cdot \underline{n}\, (\nabla^2 w)\, ds - 2\int_{\partial V} \nabla(\nabla^2 w)\underline{n}\, v \, ds = 0$$

$\Rightarrow$ Eqn is $\nabla\cdot\nabla(\nabla^2 w) = \nabla^4 w = 0$ (by the usual arguments)

Possible boundary condition:

$$\left.\begin{aligned} \nabla(\nabla^2 w)\underline{n} &= 0 \\ \nabla^2 w &= 0 \end{aligned}\right] \quad \text{Neuman}$$

$$\left.\begin{aligned} \nabla \underline{w}\cdot\underline{n} &= 0 \\ w &= 0 \end{aligned}\right] \quad \text{Dirichlet}$$

$\Rightarrow \nabla v \cdot n = 0$

$v = 0$

Need to choose b.cs such that boundary terms vanish.

Part (a) of this question was well answered, and Part (c) was generally well answered. For Part (c), some candidates failed to apply integration by parts twice. Few candidates were successful with Part (b) as most did not identify that moment equilibrium of a body can be expressed using the cross product, i.e.

$$\int_{\partial V} \boldsymbol{t}\times\boldsymbol{x}\, dS = 0.$$

Q2 a) $$-\nabla\cdot\kappa\nabla u = f$$

Multiply by ~~A~~ a test function v and integrate by parts,

$$-\int_V v(\nabla\cdot\kappa\nabla u)\,dV = \int_V v f\,dV$$

$$\rightarrow \int_V \nabla v\cdot\kappa\nabla \underline{u} = \int_V v f\,dV + \underbrace{\int_{\partial V} v\,\kappa\nabla u\cdot \underline{n}\,dS=0}_{=0}$$

$=0$ since $v=0$ when u is prescribed (b.c.)

b) $$J = \frac{1}{2}\int_V \kappa\nabla v\cdot\nabla v - v f\,dV$$

Check:

$$D_w J(v)[w] = \int_V (\kappa\nabla v)\cdot\nabla w - w f\,dV = 0$$

$\Rightarrow$ consistent with weak form in (a) $(v\to u,\ w\to v)$

c) i) Constraint: $+\nabla\cdot\kappa\nabla u + f = 0$ (1)

Control: $(u-z)^2$ (2)

Control variable: κ

Use Langrange multiplier to enforce (1), minimise (2) & add 'control' of κ for stability:

$$I = \int_V \lambda(\nabla\cdot\kappa\nabla u + f) + \frac{1}{2}(u-z)^2 + \frac{\alpha}{2}\kappa^2\,dV$$

$\frac{1}{2}$ is optional, but is conventional

α: +ve and can be varied

Q2 c) ii) To determine the equations, take directional derivative of I w.r.t. each variable (λ, u, k) and set equal to zero.

$$D_\lambda I(\cdot)[\bar{\lambda}] = \int_V \bar{\lambda}(\nabla \cdot k\nabla u + f)\, dV = 0 \quad (1)$$

$$D_u I(\,)[\bar{u}] = \int_V \lambda \nabla \cdot k \nabla \bar{u} + (u-z)\bar{u}\, dV = 0 \quad (2)$$

$$D_k I(\,)[\bar{k}] = \int_V \lambda \nabla \cdot \bar{k} \nabla u + \alpha k \bar{k}\, dV = 0 \quad (3)$$

$\bar{\lambda}, \bar{u}, \bar{k}$ are 'arbitrary'. Follow usual arguments to identify PDEs associated with (1), (2), (3).

(1) $\rightarrow \quad \nabla \cdot k \nabla u + f = 0$

(2) Need to apply integration by parts twice,

$$-\int_V (\nabla \lambda\, k) \cdot \nabla \bar{u}\, dV + \int_{\partial V} \lambda k \nabla \bar{u} \cdot n\, dS + \int_V (u-z)\bar{u}\, dV$$

Int. by parts again,

$$\int_V \nabla \cdot (k \nabla \lambda)\bar{u}\, dV - \int_{\partial V} (k \nabla \lambda \cdot n)\bar{u}\, dS + \int_{\partial V} \lambda k \nabla \bar{u} \cdot n\, dS + \int_V (u-z)\bar{u}\, dV = 0$$

$$\Longrightarrow \quad \nabla \cdot (k \nabla \lambda) + (u - z) = 0$$

b.c.'s $\begin{cases} u = 0 \text{ (prescribed in question)} \rightarrow \bar{u} = 0 \\ \text{Need } \lambda = 0 \text{ to cancel boundary term} \end{cases}$

(3) Apply integration by parts,

$$-\int_V \nabla \lambda \cdot \bar{k} \nabla u\, dV + \int_{\partial V} \lambda \bar{k} \nabla u \cdot n\, dS + \int_V \alpha k \bar{k}\, dV = 0$$

$$\rightarrow \quad -\nabla \lambda \cdot \nabla u + \alpha k = 0$$

($\lambda = 0$ on $\partial V \rightarrow$ boundary term cancels)

Q2 c) ii) cont.

Summary of eqns

(1) $\nabla \cdot \kappa \nabla u + f = 0$

(2) $\nabla \cdot (\kappa \nabla \lambda) + (u - z) = 0$

(3) $-\nabla \lambda \cdot \nabla u + \alpha \kappa = 0$

Eqns are coupled.

Note: equations are nonlinear due to products of the unknown functions (u, λ, κ).

This was not a popular question, despite its similarity to a question on the 2010 paper and to an example presented during the lectures. Of the candidates that attempted it, it was generally completed either very well or very poorly. Some candidates included control of the source term f in the functional, rather than the control variable κ.

Q5

(a) $B^2 - AC = (b^2 - ac)\,J^2$

$$J = \begin{vmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \xi}{\partial y} \\ \frac{\partial \eta}{\partial x} & \frac{\partial \eta}{\partial y} \end{vmatrix} \neq 0$$

$\Rightarrow$ A change of variable for which $J \neq 0$ leaves unchged the sign of the expression $B^2 - AC$ and $b^2 - ac$.

$\Rightarrow$ The type of a second order PDE dose not change under coordinate transformations.

(b) (i) $a = 4$, $b = 2$, $c = 1$ $\Rightarrow$ Discriminant $\Delta = b^2 - ac = 0$

The equation is parabolic. The family of characteristic curve becomes $\frac{dy}{dx} = \frac{b}{a} = \frac{1}{2} \Rightarrow y = \frac{1}{2}x + \xi$ or $\boxed{\xi = y - \frac{x}{2}}$.

The other coordinate variable η can be chosen arbitrarily, for simplicity, set $\boxed{\eta = x}$.

(ii) $a = 1$, $b = 1$, $c = \sin^2 x$ $\Rightarrow$ Discriminant $\Delta = 1 - \sin^2 x = \cos^2 x$.

$\Delta > 0 \Rightarrow$ Hyperbolic Eqn.

The characteristic equations are

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{a} = 1 \pm \cos x$$

When integrated to find the equations of the characteristic curves, they lead to the results

$$y = x + \sin x + \xi \quad \text{and} \quad y = x - \sin x + \eta$$

or

$$\boxed{\xi = y - x - \sin x} \quad \text{and} \quad \boxed{\eta = y - x + \sin x}$$

(b) (iii) $a=1,\ b=\sin x,\ c=2-\cos^2 x \Rightarrow \Delta = \sin^2 x - 2 + \cos^2 x = -1 < 0$

$\Rightarrow$ Elliptic Eqn.

Two complex characteristic equations are:

$$\frac{dy}{dx} = \frac{b \pm \sqrt{\Delta}}{a} = -\sin x \pm i$$

Integrating these complex characteristic equations gives

$$y = \cos x - ix + \xi + i\eta \quad \text{and} \quad y = \cos x + ix + \xi - i\eta$$

Taking the real and imaginary parts, we obtain

$$\boxed{\xi = y - \cos x} \quad \text{and} \quad \boxed{\eta = x}$$

(c) Use the coordinate transformation obtained in (b) (iii)

Derivatives: $\frac{\partial \xi}{\partial x} = \sin x,\ \frac{\partial \eta}{\partial x} = 1,\ \frac{\partial^2 \xi}{\partial x^2} = \cos x,\ \frac{\partial^2 \xi}{\partial x^2} = 0,\ \frac{\partial^2 \xi}{\partial x \partial y} = 0$

$\frac{\partial \xi}{\partial y} = 1,\ \frac{\partial \eta}{\partial y} = 0,\ \frac{\partial^2 \xi}{\partial y^2} = 0,\ \frac{\partial^2 \eta}{\partial y^2} = 0,\ \frac{\partial^2 \eta}{\partial x \partial \eta} = 0$

Chain rule: $$\frac{\partial^2 u}{\partial x^2} = (\sin x)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\sin x \frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} + \cos x \frac{\partial u}{\partial \xi}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \sin x \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 u}{\partial \xi^2}$$

Substituting these results into the original PDE gives:

$$(\sin x)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\sin x \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial \eta^2} + \cos x \frac{\partial u}{\partial \xi} - 2\sin x\left(\sin x \frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \xi \partial \eta}\right)$$

$$+ (2 - \cos^2 x)\frac{\partial^2 u}{\partial \xi^2} + u = 0$$

or $$\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + \cos x \frac{\partial u}{\partial \xi} + u = 0$$

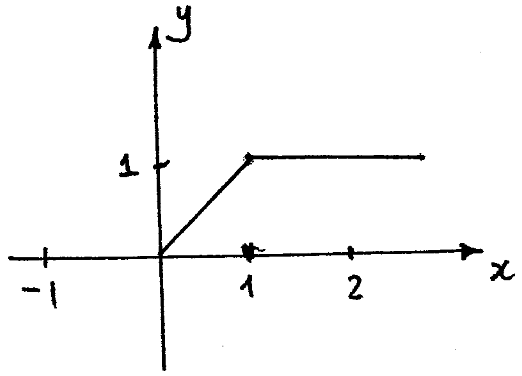
but $x = \eta$, $\Rightarrow$ $$\boxed{\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} \mp -\cos \eta \frac{\partial u}{\partial \xi} - u}$$

Q3 Examiner's comment:

This was a straightforward question and was completed well by almost all candidates. A small number of candidates did not add a complex constant when integrating the complex equation for the elliptic case.

Q 4

Initial condition



(a) Burgers' equation

(i) characteristic curves $\frac{dx}{dt} = \rho =$ constant

characteristics are straight lines

There are three cases:

~~① $x > 1+t$~~

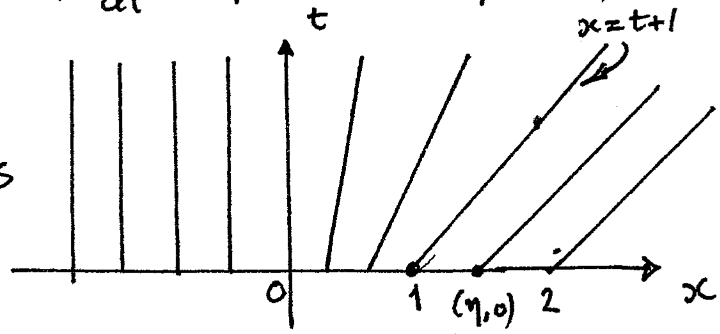
① characteristics started from $(\eta, 0)$, $\eta > 1$

$$\frac{dx}{dt} = \rho(\eta, 0) = 1 \Rightarrow x = t + \eta$$

② $\eta < 0$, $\frac{dx}{dt} = \rho(\eta, 0) = 0 \Rightarrow x = \eta$

③ $0 \leq \eta \leq 1$, $\frac{dx}{dt} = \eta \Rightarrow x = \eta(1+t)$

Characteristics



Hence ① For $x > 1+t \Rightarrow \rho(x,t) = 1$

② For $x < 0 \Rightarrow \rho(x,t) = 0$

③ For $0 \leq x \leq 1+t$, $0 \leq \eta \leq 1$

$$\left.\begin{array}{l} x = \eta(1+t) \\ \rho(x,t) = \rho(\eta, 0) = \eta \end{array}\right\} \Rightarrow x = \rho(x,t)(1+t)$$

$$\Rightarrow \boxed{\rho(x,t) = \frac{x}{1+t}}$$

(ii) characteristics diverge $\Rightarrow$ no shock (discontinuity) forms.

At $t = 1$

$$\begin{cases} ③\ \rho(x,1) = 0 & \text{for } x < 0 \\ ②\ \rho(x,1) = x/2 & \text{for } 0 \leq x \leq 2 \\ ①\ \rho(x,1) = 1 & \text{for } x > 2 \end{cases}$$

(b) Traffic flow $\frac{\partial \rho}{\partial t} + (1-2\rho)\frac{\partial \rho}{\partial x} = 0$

(i) characteristic Eqn.

$\frac{dx}{dt} = (1-2\rho) = \text{constant} \Rightarrow$ characteristic are straight lines.

There are three cases

① initial point $(\xi, 0)$ such as $\xi > 1$, $\rho(\xi,0)=1$

Charact; $\frac{dx}{dt} = 1-2 = -1 \Rightarrow x = \xi - t$, or $\boxed{x+t=\xi} \Rightarrow \boxed{\rho(x,t)=1}$

② $\xi < 0$, $\rho(\xi,0)=0$, characteristics are

$\frac{dx}{dt} = 1 \Rightarrow x = \xi + t$, or $\boxed{x-t=\xi} \Rightarrow \boxed{\rho(x,t)=0}$

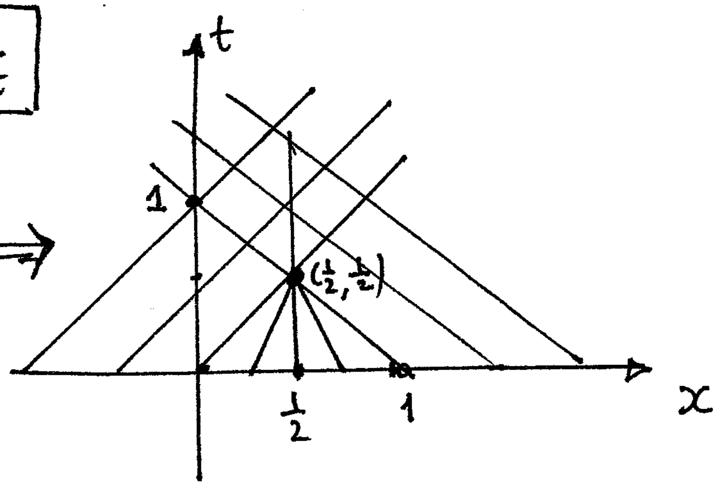
③ for $0 \le \xi \le 1$, $\rho(\xi,0) = \xi$, characteristics are

$\frac{dx}{dt} = (1-2\xi) \Rightarrow x = (1-2\xi)t + \xi$

However on this characteristic, $\rho(x,t) = \rho(\xi,0) = \xi$ $\Bigg\} \Rightarrow$

$\boxed{\rho(x,t) = \frac{x-t}{1-2t}}$

Characteristics $\Rightarrow$



Hence for

① $x+t > 1$, $\rho(x,t)=1$

② $x-t < 0$, $\rho(x,t)=0$

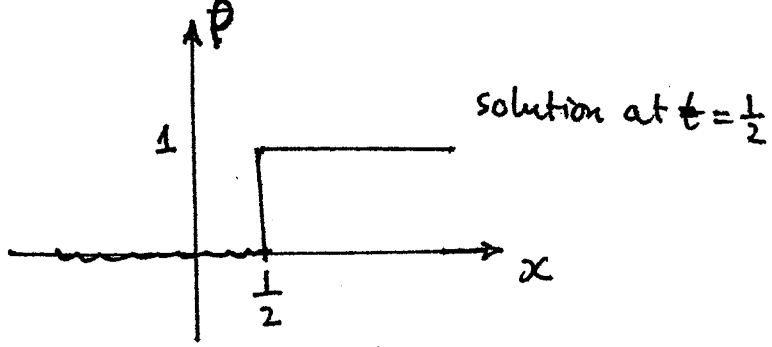
③ $x-t \ge 0$ and $x+t \le 1$, $\rho(x,t) = \frac{x-t}{1-2t}$, valid for $t < \frac{1}{2}$.

(b) (ii) Characteristics converges. Discontinuity forms at $x = \frac{1}{2}$, $t = \frac{1}{2}$

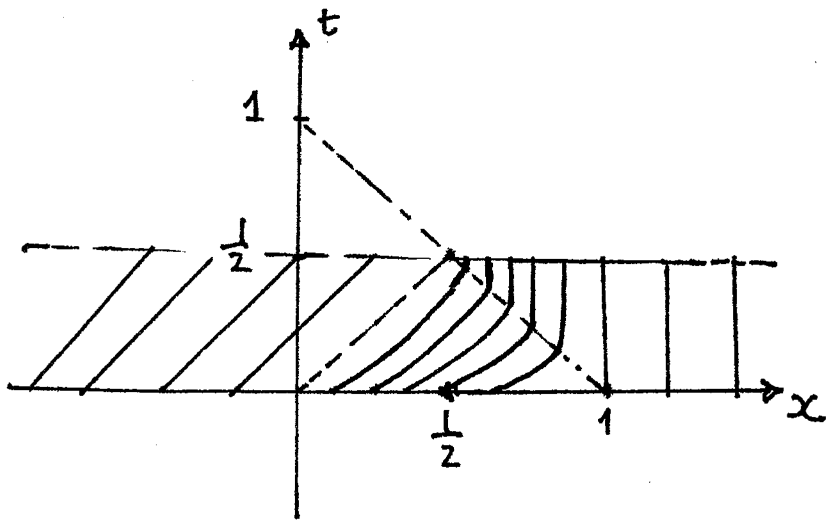
Q4 cont

At $t = \frac{1}{2}$, $\rho(x, \frac{1}{2}) = 0$ for $x < \frac{1}{2}$

$\rho(x, \frac{1}{2}) = 1$ for $x \geq \frac{1}{2}$.



(iii) The characteristics have been shown in (b) (i).
The speed of car is $u = 1 - \rho$.
The car trajectories are sketched as below



This question was completed well by most candidates. A common error was to not include time t when defining the three regions of interest, and to express the domains only in terms of x (which is correct only when $t = 0$).

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