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PART IIB
2011

4M13: COMPLEX ANALYSIS AND OPTIMIZATION

1. In each case the complex integrand has a branch point at $z = 0$, so the contour is the same in each case, the standard "keyhole" contour.

   a) Let us call the integral to be evaluated I. Consider the complex integral,

   $$J = \int \frac{\log z}{1+z^2}\, dz$$

   On a small circle of radius ε around the origin, write $z = \varepsilon e^{i\theta}$, so the integral becomes

   $$\int_0^{2\pi} d\theta \frac{\log \varepsilon + i\theta}{1+\varepsilon^2 e^{i2\theta}} i\varepsilon e^{i\theta} \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

   Similarly on a very large circle of radius R, $z = \mathrm{R}e^{i\theta}$,

   $$\int_0^{2\pi} d\theta \frac{\log R + i\theta}{1+R^2 e^{i2\theta}} i R e^{i\theta} \to 0 \quad \text{as} \quad R \to \infty.$$

   What remains are the straight sections just above and below the positive real axis,

   $$J = \int_0^{\infty} dx \frac{\log x}{1+x^2} + \int_{\infty}^{0} dx \frac{\log x + 2\pi i}{1+x^2} = -2\pi i I$$

   Now due to the residue theorem, the complex integral is $2\pi i \times$ the sum of the residues of the poles inside the contour. There are two simple poles at $z = \pm i$, where the residues are $\lim_{z\to i} (z-i)/(z-i)(z+i) \times \log z = 1/2i \times \pi i/2 = \pi/4$ and $\lim_{z\to -i} (z+i)/(z-i)(z+i) \times \log z = -1/2i \times 3\pi i/2 = -3\pi/4$, so

   $$I = 2\pi i(\pi/4 - 3\pi/4)/(-2\pi i) = \pi/2$$ [40%]

   b) Again let us call the integral I, and consider the complex integral

   $$J = \int \frac{(\log z)^2}{1+z^2}\, dz$$

   The contour and the integrals on the circular sections are the same as in part a). On the straight sections, we have

   $$\begin{aligned} J &= \int_0^{\infty} dx \frac{(\log x)^2}{1+x^2} + \int_{\infty}^{0} dx \frac{(\log x + 2\pi i)^2}{1+x^2} \\ &= \int_0^{\infty} dx \frac{4\pi^2 - 4\pi i \log x}{1+x^2} = 4\pi^2 \frac{\pi}{2} - 4\pi i I = 2\pi^3 - 4\pi i I \end{aligned}$$

   where we have used the result of a) to evaluate one of the integrals. The value of J again is obtained from the residue theorem,

   $$J = 2\pi i \left(\frac{(\log i)^2}{2i} + \frac{(\log -i)^2}{-2i} \right) = \pi \left(-\frac{\pi^2}{4} + \frac{9\pi^2}{4} \right) = 2\pi^3$$

   So therefore $I = 0$. [30%]

c) Call the integral I again, and consider the complex integral

$$J=\int \frac{(\log z)^3}{1+z^2}dz$$

The integral vanishes on the circular sections as in parts a) and b), and on the straight sections we have

$$\begin{aligned} J &= \int_0^\infty dx\,\frac{(\log x)^3}{1+x^2}+\int_\infty^0 dx\,\frac{(\log x+2\pi i)^3}{1+x^2} \\ &= -\int_0^\infty dx\,\frac{6\pi i(\log x)^2-12\pi^2\log x-8\pi^3 i}{1+x^2} \\ &= 4\pi^4 i-6\pi i I \end{aligned}$$

where we have used the results from parts a) and b). The residue theorem gives

$$J=2\pi i\left(\frac{(\log i)^3}{2i}+\frac{(\log -i)^3}{-2i}\right)=\pi\left(-\frac{i\pi^3}{8}+\frac{27i\pi^3}{8}\right)=\frac{13}{4}i\pi^4$$

So $I=(4\pi^4 i-\frac{13}{4}i\pi^4)/6\pi i=\pi^3/8$. [30%]

Most candidates understood the basics of contour integration, although Jordan's Lemma was misapplied. Part (a) was well done by almost all, but few were able to successfully tackle parts (b) and (c).

2.

a) The Fourier Transform is given by

$$\begin{aligned} F(\omega) &= \int_{-\infty}^{\infty} dx\, f(x)\, e^{i\omega x} \\ &= \int_{0}^{\infty} dx\, e^{-ax} e^{i\omega x} = \int_{0}^{\infty} dx\, e^{(i\omega - a)x} \\ &= \frac{1}{i\omega - a}\Big[e^{(i\omega - a)x} \Big]_{x=0}^{x=\infty} = \frac{1}{i\omega - a} \end{aligned}$$

[20%]

b) The inverse transform gives $f(x)$ in terms of an integral of $F(\omega)$,

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, F(\omega) e^{-i\omega x} \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{e^{-i\omega x}}{i\omega - a} \end{aligned}$$

Consider first $x < 0$, and let us complete the contour in the upper half plane by a large semicircle, so that Jordan's Lemma applies. The pole of the integrand is at $\omega = -ia$, so there are no poles in the closed contour, i.e. $f(x) = 0$ as it should be. For $x > 0$, complete the contour in the lower half plane. The integral vanishes on the large semicircle again due to Jordan's Lemma. The integral then is $2\pi i \times$ the residue at the pole, so that

$$f(x) = \frac{1}{2\pi} \lim_{\omega \to -ia} 2\pi(\omega + ia)/(\omega + ia) \times e^{-i\omega x} = e^{-ax}$$

[80%]

A popular and easy question that was well done by most candidates. Some candidates used Jordan's Lemma as a throw-away remark without explaining their working. And some considered only x>0, without considering the case x<0.

Question 3

(a) The Taylor series expansion is:

$$f(x_{k+1}) = f(x_k) + (x_{k+1} - x_k)f'(x_k) + \tfrac{1}{2}(x_{k+1} - x_k)^2 f''(x_k) + R$$

where R represents the higher order terms.

Let $R = 0$, i.e. approximate $f(x_{k+1})$ as a quadratic.

Differentiating with respect to x_{k+1}:

$$f'(x_{k+1}) = f'(x_k) + (x_{k+1} - x_k)f''(x_k)$$

If x_{k+1} is the location of the minimum of $f(x)$, then $f'(x_{k+1}) = 0$, and therefore:

$$0 = f'(x_k) + (x_{k+1} - x_k)f''(x_k)$$

$$\therefore \quad x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$ [15%]

(b) (i)

The variable costs are those of the tower and of the cables.

The volume of the tower is: $V_T = hA_T$

By simple trigonometry: $\tan\theta = \dfrac{h}{d} \Rightarrow h = d\tan\theta$

$$V_T = A_T d\tan\theta = \frac{Wd\tan\theta}{\sigma_{0T}}$$

The volume of the cable used is: $V_C = 2lA_C$

where l is the length of one of the two cables.

By simple trigonometry: $\cos\theta = \dfrac{d}{l} \Rightarrow l = \dfrac{d}{\cos\theta}$

$$\therefore \quad V_C = \frac{2dA_C}{\cos\theta} = \left(\frac{2d}{\cos\theta}\right)\left(\frac{W}{2\sigma_{0C}\sin\theta}\right) = \frac{Wd}{\sigma_{0C}\sin\theta\cos\theta}$$

$$C_{\text{section}} = C_C V_C + C_T V_T$$

$$\therefore \quad C_{\text{section}} = C_C \frac{Wd}{\sigma_{0C}\sin\theta\cos\theta} + C_T \frac{Wd\tan\theta}{\sigma_{0T}}$$

$$\therefore \quad C_{\text{section}} = \frac{C_C Wd}{\sigma_{0C}}\left[\frac{1}{\sin\theta\cos\theta} + \frac{C_T\sigma_{0C}}{C_C\sigma_{0T}}\tan\theta\right]$$

Using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$

$$\frac{1}{\sin\theta\cos\theta} = \frac{2}{\sin 2\theta} = 2\csc 2\theta$$

$$\therefore \quad C_{\text{section}} = \frac{C_C Wd}{\sigma_{0C}}(2\csc 2\theta + \alpha\tan\theta) \text{ where } \alpha = \frac{C_T\sigma_{0C}}{C_C\sigma_{0T}}$$ [20%]

(ii)

$$f(\theta) = 2\csc 2\theta + \alpha \tan\theta$$

$$\therefore \quad f'(\theta) = -4\csc 2\theta \cot 2\theta + \alpha \sec^2\theta$$

$$\therefore \quad f''(\theta) = -4(-2\csc 2\theta \cot 2\theta)\cot 2\theta - 4\csc 2\theta(-2\csc^2 2\theta) + 2\alpha \sec^2\theta \tan\theta$$

$$\therefore \quad f''(\theta) = 8\csc 2\theta \cot^2 2\theta + 8\csc^3 2\theta + 2\alpha \sec^2\theta \tan\theta$$

$$\therefore \quad f''(\theta) = 8\csc^3 2\theta\left(1+\cos^2 2\theta\right) + 2\alpha \sec^2\theta \tan\theta$$

$$\theta_{k+1} = \theta_k - \frac{f'(\theta_k)}{f''(\theta_k)}$$

Hence:

θ_k	$f'(\theta_k)$	$f''(\theta_k)$	$\theta_{k+1} = \theta_k - \frac{f'(\theta_k)}{f''(\theta_k)}$
0.5	–2.403015684	18.05578556	0.633088404
0.633088404	–0.549041959	11.17309363	0.682228058
0.682228058	–0.025464505	10.23804534	0.684715301

Thus $\theta_{\text{opt}} \approx 0.6847$ radians [45%]

(iii) At the minimum $f'(\theta) = 0$ and $f''(\theta) > 0$.

$$f'(\theta) = -4\csc 2\theta \cot 2\theta + \alpha \sec^2\theta = 0$$

$$\therefore \quad 0 = -4\frac{1}{\sin 2\theta}\frac{\cos 2\theta}{\sin 2\theta} + \alpha\frac{1}{\cos^2\theta}$$

$$\therefore \quad \frac{\alpha}{\cos^2\theta} = \frac{4\cos 2\theta}{\sin^2 2\theta} = \frac{4(1-2\sin^2\theta)}{(2\sin\theta\cos\theta)^2}$$

using $\cos 2\theta = 1 - 2\sin^2\theta$ and $\sin 2\theta = 2\sin\theta\cos\theta$.

$$\therefore \quad \alpha = \frac{(1-2\sin^2\theta)}{\sin^2\theta}$$

$$\therefore \quad \sin^2\theta = \frac{1}{2+\alpha} \quad \Rightarrow \quad \theta = \sin^{-1}\left(\frac{1}{\sqrt{2+\alpha}}\right)$$

For $\alpha = 0.5 \quad \therefore \quad \theta_{\text{opt}} = \sin^{-1}\left(\frac{1}{\sqrt{2.5}}\right) = \sin^{-1}(0.6324555) = 0.68472$ radians

Should check $f''(\theta) > 0$.

From above $\quad f''(\theta) = 8\csc^3 2\theta\left(1+\cos^2 2\theta\right) + 2\alpha \sec^2\theta \tan\theta$

For $\theta = 0.68472$ radians $\quad f''(\theta) = 10.2062 \quad \Rightarrow \quad$ a minimum

Thus Newton's method has found a good estimate of the true minimum in three iterations. This implies that $f(\theta)$ is well approximated as a quadratic function. [20%]

A popular question that was well done by Part IIB candidates but rather less well done by Part IIA candidates. Most candidates seemed to know what they were trying to do. The most common sources of error were poor differentiation skills and poor calculator use.

Question 4

(a) The objective is to minimise r_o.

The constraint on yielding $\sigma_y \pi (r_o^2 - r_i^2) \geq P$ can be rewritten as $(r_o^2 - r_i^2) \geq \frac{P}{\pi \sigma_y}$, so in standard form: $g_1 = \frac{P}{\pi \sigma_y} - (r_o^2 - r_i^2) \leq 0$

The maximum deflection constraint $\delta_{max} = \frac{5 \rho g L^4}{96 E (r_o^2 + r_i^2)} \leq 0.001L$ can be rewritten as

$\frac{625 \rho g L^3}{12E} \leq (r_o^2 + r_i^2)$, so in standard form: $g_2 = \frac{625 \rho g L^3}{12E} - (r_o^2 + r_i^2) \leq 0$

In addition, it is physically obvious that $r_o \geq r_i$ and $r_i \geq 0$. Thus, in standard form the task is

Minimise r_o

Subject to $g_1 = K_1 - (r_o^2 - r_i^2) \leq 0$ where $K_1 = \frac{P}{\pi \sigma_y}$

$g_2 = K_2 - (r_o^2 + r_i^2) \leq 0$ where $K_2 = \frac{625 \rho g L^3}{12E}$

$g_3 = r_i - r_o \leq 0$ i.e. $r_o \geq r_i$

$g_4 = -r_i \leq 0$ i.e. $r_i \geq 0$ [15%]

(b) For the values given

$$K_1 = \frac{P}{\pi \sigma_y} = \frac{50 \times 10^3}{\pi \times 200 \times 10^6} = 7.9577 \times 10^{-5} \text{ m}^2$$

$$K_2 = \frac{625 \rho g L^3}{12E} = \frac{625 \times 77000 \times 2.5^3}{12 \times 200 \times 10^9} = 3.1331 \times 10^{-4} \text{ m}^2$$

Thus the equations of the constraints when $g = 0$ are:

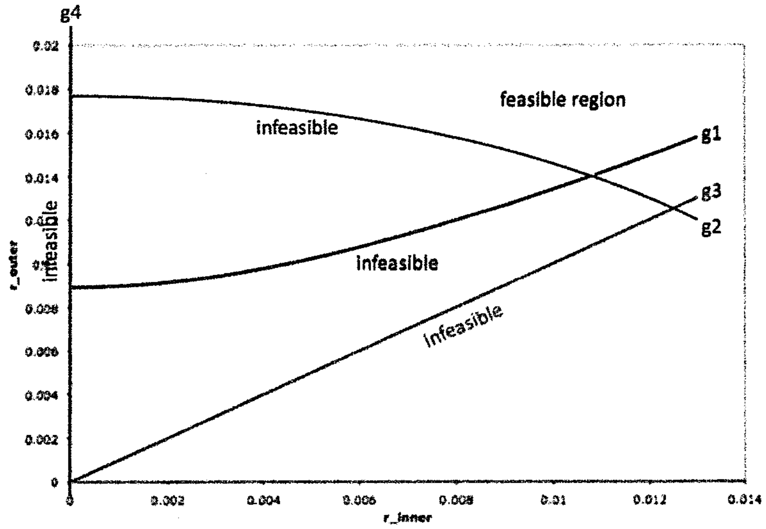
$$g_1: \quad r_o = \sqrt{7.9577 \times 10^{-5} + r_i^2}$$

$$g_2: \quad r_o = \sqrt{3.1331 \times 10^{-4} - r_i^2}$$

$$g_3: \quad r_o = r_i$$

The feasible region is therefore as shown on the next page.

Contours of the objective function are horizontal lines. Thus, by inspection the optimum occurs at the intersection of g_1 and g_2. So these two constraints are active, and g_3 and g_4 are inactive.



g_1 and g_2 intersect when

$$r_o = \sqrt{7.9577\times10^{-5} + r_i^2} = \sqrt{3.1331\times10^{-4} - r_i^2}$$

$$\therefore\ 2r_i^2 = 3.1331\times10^{-4} - 7.9577\times10^{-5}$$

$$\therefore\ r_i = 0.01081\,\text{m}$$

$$\therefore\ r_o = 0.01402\,\text{m}$$

[35%]

(c) From (b) we know g_3 and g_4 are inactive at the optimum. Thus

$$L = r_o + \mu_1\left[K_1 - \left(r_o^2 - r_i^2\right)\right] + \mu_2\left[K_2 - \left(r_o^2 + r_i^2\right)\right]$$

$$\therefore\ \frac{\partial L}{\partial r_o} = 1 + \mu_1[-2r_o] + \mu_2[-2r_o] = 0 \quad (1)$$

$$\therefore\ \frac{\partial L}{\partial r_i} = \mu_1[2r_i] + \mu_2[-2r_i] = 0 \quad (2)$$

$$\mu_1\left[K_1 - \left(r_o^2 - r_i^2\right)\right] = 0 \quad (3)$$

$$\mu_2\left[K_2 - \left(r_o^2 + r_i^2\right)\right] = 0 \quad (4)$$

Case (i) $\mu_1 = 0$ and $\mu_2 = 0$

$(1) \Rightarrow\ 1 = 0$

$\therefore$ **impossible**

Case (ii) $\mu_1 = 0$ and $\mu_2 > 0$

$(2) \Rightarrow\ -2\mu_2 r_i = 0 \ \Rightarrow\ r_i = 0$

$(4) \Rightarrow\ K_2 - r_o^2 = 0 \ \Rightarrow\ r_o = \sqrt{K_2} = 0.01770\,\text{m}$

Need to check $g_1 = K_1 - \left(r_o^2 - r_i^2\right) \le 0$ is not violated:

$g_1 = 7.9577\times10^{-5} - \left(3.1331\times10^{-4} - 0^2\right) = -2.3373\times10^{-4} \le 0 \ \therefore$ OK

$(1) \Rightarrow \quad 1-2\mu_2 r_o = 0 \quad \Rightarrow \quad \mu_2 = \dfrac{1}{2r_o} = 28.25\ \text{m}^{-1}$

∴ a minimum

Case (iii) $\mu_1 > 0$ and $\mu_2 = 0$

$(2) \Rightarrow \quad 2\mu_1 r_i = 0 \quad \Rightarrow \quad r_i = 0$

$(3) \Rightarrow \quad K_1 - r_o^2 = 0 \quad \Rightarrow \quad r_o = \sqrt{K_1} = 0.00892\ \text{m}$

Need to check $g_2 = K_2 - \left(r_o^2 + r_i^2\right) \le 0$ is not violated:

$g_2 = 3.1331\times10^{-4} - \left(7.9577\times10^{-5} + 0^2\right) = 2.3373\times10^{-4}$ ∴ g_2 is violated

∴ impossible

Case (iv) $\mu_1 > 0$ and $\mu_2 > 0$

$(3) \Rightarrow \quad K_1 - \left(r_o^2 - r_i^2\right) = 0$

$(4) \Rightarrow \quad K_2 - \left(r_o^2 + r_i^2\right) = 0$

$(3)+(4) \Rightarrow \quad K_1 + K_2 - 2r_o^2 = 0 \quad \Rightarrow \quad r_o = \sqrt{\dfrac{K_1 + K_2}{2}} = 0.01402\ \text{m}$

$(4)-(3) \Rightarrow \quad K_2 - K_1 - 2r_i^2 = 0 \quad \Rightarrow \quad r_i = \sqrt{\dfrac{K_2 - K_1}{2}} = 0.01081\ \text{m}$

$(2) \Rightarrow \quad 2\mu_1 r_i - 2\mu_2 r_i = 0 \quad \Rightarrow \quad \mu_1 = \mu_2$

$(1) \Rightarrow \quad 1 - 2\mu_1 r_o - 2\mu_2 r_o = 0 \quad \Rightarrow \quad 1 - 4\mu_1 r_o = 0 \quad \Rightarrow \quad \mu_1 = \mu_2 = \dfrac{1}{4r_o} = 17.84\ \text{m}^{-1}$

∴ a minimum

As the value of the objective (r_o) is smaller than for the minimum in case (ii) this is the global minimum. [40%]

(d) The values of the K-T multipliers give the sensitivity of the objective function to changes in the constraint limits. As $\mu_1 = \mu_2$ the objective function will change at the same rate as the associated constraint limits (the values of K_1 and K_2) are relaxed.

This is also apparent from the result for r_o found above – $r_o = \sqrt{\dfrac{K_1 + K_2}{2}}$. [10%]

A popular and well done question for Part IIB candidates; less popular and less well done for Part IIA candidates. There were a few instances of candidates not knowing how to apply the Kuhn-Tucker multiplier method, but the most common source of error was a failure to identify the feasible region correctly. Candidates who got this wrong and then tried to fudge the rest of the answer in line with this inevitably found the going tough.