ENGINEERING TRIPOS PART IIB

Monday 2 May 2011 9 to 10.30

Module 4A8

ENVIRONMENTAL FLUID MECHANICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Attachments: Data sheets (7 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Show that the dry adiabatic lapse rate (DALR) is given by:

$$\frac{dT}{dz} = -\frac{g}{c_p}$$

Sketch the vertical temperature distribution for various atmospheric stability types. [30%]

(b) The potential temperature is defined as $\theta = T\,(p_o/p)^{\kappa}$, where T is the temperature, p is the pressure, p_o is a reference pressure, $\kappa = R/c_p$, and R is the gas constant.

(i) Using the given definition for θ, deduce stability conditions for a density-stratified atmosphere on a calm day. [40%]

(ii) If the temperature variation with z is given by:

$$T(z) = A + Be^{-z/H}$$

what is the necessary condition for this variation to be gravitationally stable when A, B and H are positive constants? [30%]

2 Consider a city located in a valley as in Fig. 1. On a clear day, the mountain slopes are heated by sunshine and this sets up convective motions.

(a) Draw a clear sketch of a simple model flow that can be used to analyze the convective flow down the slopes. Mark all the relevant physical processes and quantities in your diagram clearly. [30%]

(b) Select an appropriate control volume in the model flow and assume this flow to be steady. Deduce differential equations for conservation of mass, momentum and energy, stating clearly any assumptions made. [40%]

(c) Assuming a power law variation with distance down the slope for the parameters describing the flow, find solutions to the variation of the air velocity and the temperature down the slope and sketch the flow pattern that emerges from this solution for day and night. [30%]

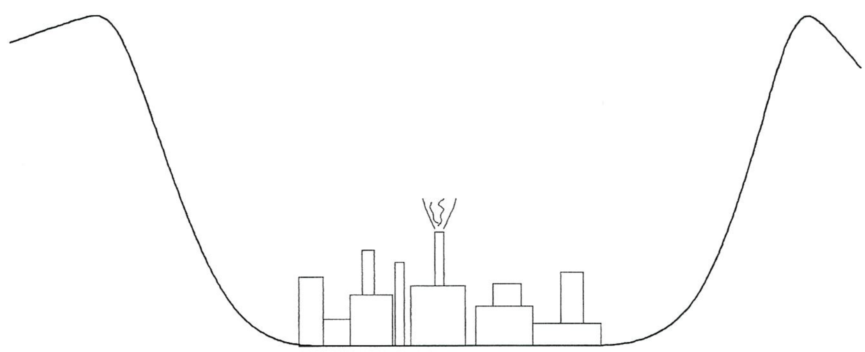


Fig. 1

3 A river has width W and depth H. The depth is changing with distance down the river as clean rainwater enters the river, but the stream velocity U and the width are assumed to stay constant. Cross-stream velocities are assumed negligible. At $x = 0$, with x the coordinate along the river, the height is H_0 and a pollutant is continuously released with the result that the water becomes polluted at a concentration $c = c_0$ (units: kg m^{-3}), assumed uniform in both cross-stream directions but not in the x–direction. The flow rate of clean rainwater entering the river is $\dot{m}'$ per unit river length (units of $\dot{m}'$: kg m^{-1} s^{-1}). The pollutant is undergoing a decomposition reaction whose reaction rate is given by $-kc$, where k is a positive constant (units: s^{-1}). The water is considered safe when the pollutant has concentration below $c_0/10$.

(a) Show that the concentration c obeys:

$$\frac{dc}{dx} = -\left(\frac{1}{H}\frac{dH}{dx} + \frac{k}{U}\right)c$$

[40%]

(b) For the two limiting cases of $\dot{m}' = 0$ with $k > 0$, and of $k = 0$ with $\dot{m}' > 0$, derive expressions for the distance downstream where the pollutant concentration falls below the safe limit. [30%]

(c) Due to erosion of the river bed, a strong stable density stratification can exist in the flow. Discuss briefly the implications of this possibility for the uniformity of concentration in the cross-stream directions that we assumed in Parts (a) and (b). If the pollutant is released from a source close to the river bed, sketch qualitatively the concentration variation with depth at a few downstream locations. [30%]

4 (a) Consider diffusion from a point source in a uniform wind of mean velocity U. If the turbulent diffusivity K is given by $K = CUL_t$, where C is a constant and L_t is the effective turbulent lengthscale causing the diffusion, explain why we may expect the dispersion coefficient σ to vary with distance x downstream of the source as x for small x, and as $x^{1/2}$ for large x. Explain also why the vertical dispersion coefficient is smaller than the horizontal one. [50%]

(b) A straight road is elevated 25 m above the ground. The traffic consists of a steady flow of cars at a rate of 1 car per second, each moving at a speed of 10 m s^{-1}. Each car emits 1 g s^{-1} of pollutant. The wind is horizontal and normal to the road direction and has a speed of 5 m s^{-1}. Assume a neutrally-stable atmosphere, urban conditions for the dispersion coefficient, and a fully-reflecting ground. For a distance 150 m downwind from the road, show that the concentration of pollutant reaches a maximum at a height of about 22 m and calculate the concentration there. [50%]

END OF PAPER

4A8: Environmental Fluid Mechanics

Part I: Turbulence and Fluid Mechanics

DATA CARD

Rotating Flows

Geostrophic Flow

$$-\frac{1}{\rho}\nabla p = 2\underline{\Omega}\times\underline{u}$$

Ekman Layer Flow

$$-2\Omega_z v = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2}$$

$$2\Omega_z u = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2}$$

$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z}$$

OR

$$-2\Omega_z v = \nu\frac{\partial^2 u}{\partial z^2}$$

$$-2\Omega_z (u_g - u) = \nu\frac{\partial^2 v}{\partial z^2}$$

GEOSTROPHIC VELOCITY

Solution is

$$u = u_g\left[1 - e^{-z/\Delta}\cos\frac{z}{\Delta}\right]$$

$$v = u_g e^{-z/\Delta}\sin\frac{z}{\Delta}$$

$$\Delta = \left(\frac{\nu}{\Omega_z}\right)^{1/2}$$

Turbulent Flows – **Incompressible**

Continuity Equation $$\nabla \bullet \underline{U} = \frac{\partial U_i}{\partial x_i} = 0$$

Momentum Equation $$\rho \frac{DU_i}{Dt} = -\frac{\partial P}{\partial x_i} + \mu \frac{\partial^2 U_i}{\partial x_j^2} + F_i$$

Enthalpy Equation $$\rho c_p \frac{DT}{Dt} = -k \frac{\partial^2 T}{\partial x_i^2}$$

Reynolds Transformation $$U_i = \overline{U_i} + u_i \quad \text{etc}$$

Reynolds Stress $= -\rho \overline{u_i u_j}$, Reynolds Heat Flux $= -\rho c_p \overline{u_j \Theta}$

Turbulent Kinetic Energy, k, Equation

$$\frac{Dk}{Dt} = -\overline{u_i u_k} \frac{\partial \overline{U_i}}{\partial x_k} - \varepsilon + \frac{\overline{f_i u_i}}{\rho} + \text{ transport of kinetic energy forms}$$

Mean temperature equation

$$\rho c_p \frac{D\overline{T}}{Dt} = \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial \overline{T}}{\partial x_j} - \overline{u_j \Theta} \rho c_p \right)$$

Temperature variance, σ, equation

$$\frac{D\sigma}{Dt} = -2\overline{u_j \Theta} \frac{\partial \overline{T}}{\partial x_j} - \varepsilon_\sigma + \text{molecular diffusion}$$

In flows with thermally driven motion

$$\frac{f_i u_i}{\rho} = \frac{g}{T} \bullet \overline{\Theta u_i}\ , \qquad i = \text{Vertical direction}$$

Dissipation of turbulent kinetic energy $$\varepsilon \approx \frac{u'^3}{\ell}$$

Kolmogorov microscale $$\eta = \left(\frac{\nu^3}{\varepsilon} \right)^{1/4}$$

Taylor microscale (λ) $$\varepsilon = 15\nu \frac{u'^2}{\lambda^2} \quad (\nu \text{ is the kinematic viscocity})$$

Density Influenced Flows

Atmospheric Boundary Layer

$$\left.\frac{dT}{dz}\right|_{\text{NEUTRAL STABILITY}} = -\frac{g}{c_p} = \left.\frac{dT}{dz}\right|_{\text{DALR}}$$

$$R_i = \frac{g}{T}\frac{\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{\text{DALR}}}{\left(\frac{dU}{dz}\right)^2} = \text{RICHARDSON NUMBER}$$

Neutral Stability

$$U = \frac{u_*}{\kappa}\ln\frac{z}{z_0}; \quad \frac{dU}{dz} = \frac{u_*}{\kappa z}$$

$$u_* = \sqrt{\frac{\tau_w}{\rho}} \quad ; \quad \kappa = \text{Von Karman Constant} = 0.40$$

Non-Neutral Stability

$$\text{L} = \text{Monin-Obukhov length} = -\frac{u_*^3}{\kappa \frac{g}{T}\frac{Q}{\rho c_p}}$$

Q = surface heat flux

$$\frac{dU}{dz} = \frac{u_*}{\kappa z}\left(1 - 15\frac{z}{L}\right)^{-1/4} \qquad \text{Unstable}$$

$$= \frac{u_*}{\kappa z}\left(1 + 4.7\frac{z}{L}\right) \qquad \text{Stable}$$

Buoyant plume for a point source

$$\frac{d}{dz}\pi R^2 w = 2\pi R u_e \qquad \text{(i)}$$

$$\frac{d}{dz}\rho\pi R^2 w = \rho_a 2\pi R u_e \qquad \text{(ii)}$$

$$\frac{d}{dz}\rho\pi R^2 w^2 = g(\rho_a - \rho)\pi R^2 \qquad \text{(iii)}$$

(i) and (iii) give

$$\pi R^2 w\left(\frac{\rho_a - \rho}{\rho_a}\right)g = \text{constant} = F_0 \text{ (buoyancy flux)}$$

$$u_e = \alpha w$$

(α = Entrainment coefficient)

$$N^2 = -\frac{g}{\rho}\frac{d\rho}{dz} = \frac{g}{T}\frac{dT}{dz}$$

Actually $$\frac{g}{T}\left(\frac{dT}{dz} - \left.\frac{dT}{dz}\right|_{\text{DALR}}\right)$$

N = Brunt – Vaisala Frequency or Buoyancy Frequency

4A8: Environmental Fluid Mechanics

Part II: Dispersion of Pollution in the Atmospheric Environment

DATA CARD

Transport equation for the mean of the reactive scalar ϕ:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_j \frac{\partial \overline{\phi}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial \overline{\phi}}{\partial x_j} \right) + \overline{\dot{w}}$$

Transport equation for the variance of the reactive scalar ϕ:

$$\frac{\partial g}{\partial t} + \overline{u}_j \frac{\partial g}{\partial x_j} = \frac{\partial}{\partial x_j}\left(K \frac{\partial g}{\partial x_j} \right) + 2K\left(\frac{\partial \overline{\phi}}{\partial x_j} \right)^2 - \frac{2}{T_{turb}} g + 2\overline{\phi' \dot{w}'}$$

Mean concentration of pollutant after instantaneous release of Q kg at t=0:

$$\overline{\phi}(x,y,z,t) = \frac{Q}{8(\pi t)^{3/2}(K_x K_y K_z)^{1/2}} \exp\left[-\frac{1}{4t}\left(\frac{(x-x_0)^2}{K_x} + \frac{(y-y_0)^2}{K_y} + \frac{(z-z_0)^2}{K_z} \right) \right]$$

Gaussian plume spreading in two dimensions from a source at $(0,0,z_0)$ emitting Q kg/s:

$$\overline{\phi}(x,y,z) = \frac{Q}{2\pi} \frac{1}{U\sigma_y \sigma_z} \exp\left[-\left(\frac{y^2}{2\sigma_y^2} + \frac{(z-z_0)^2}{2\sigma_z^2} \right) \right]$$

One-dimensional spreading from line source emitting Q/L kg/s/m :

$$\overline{\phi}(x,y) = \frac{Q}{UL} \frac{1}{\sqrt{2\pi}\sigma_y} \exp\left(-\frac{y^2}{2\sigma_y^2} \right)$$

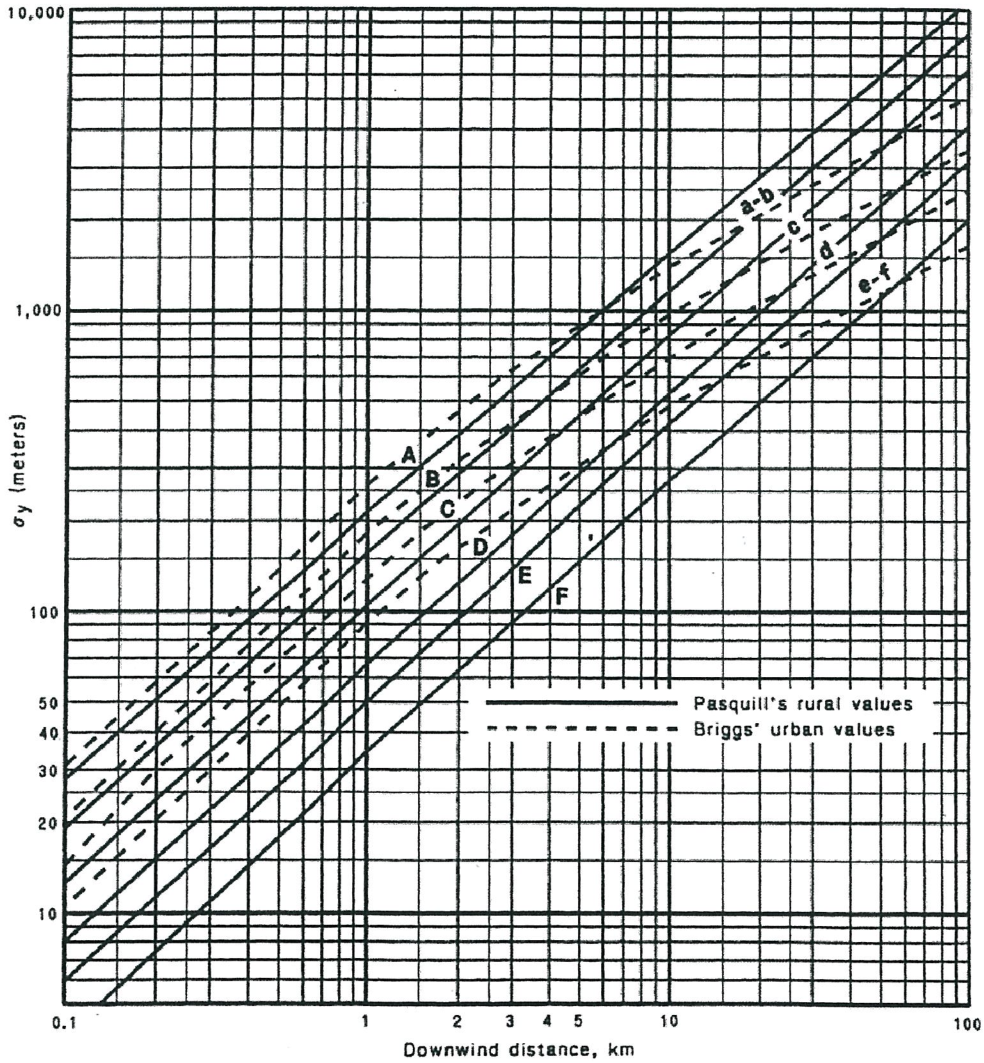
Relationship between eddy diffusivity and dispersion coefficient:

$$\sigma^2 = 2\frac{x}{U} K$$

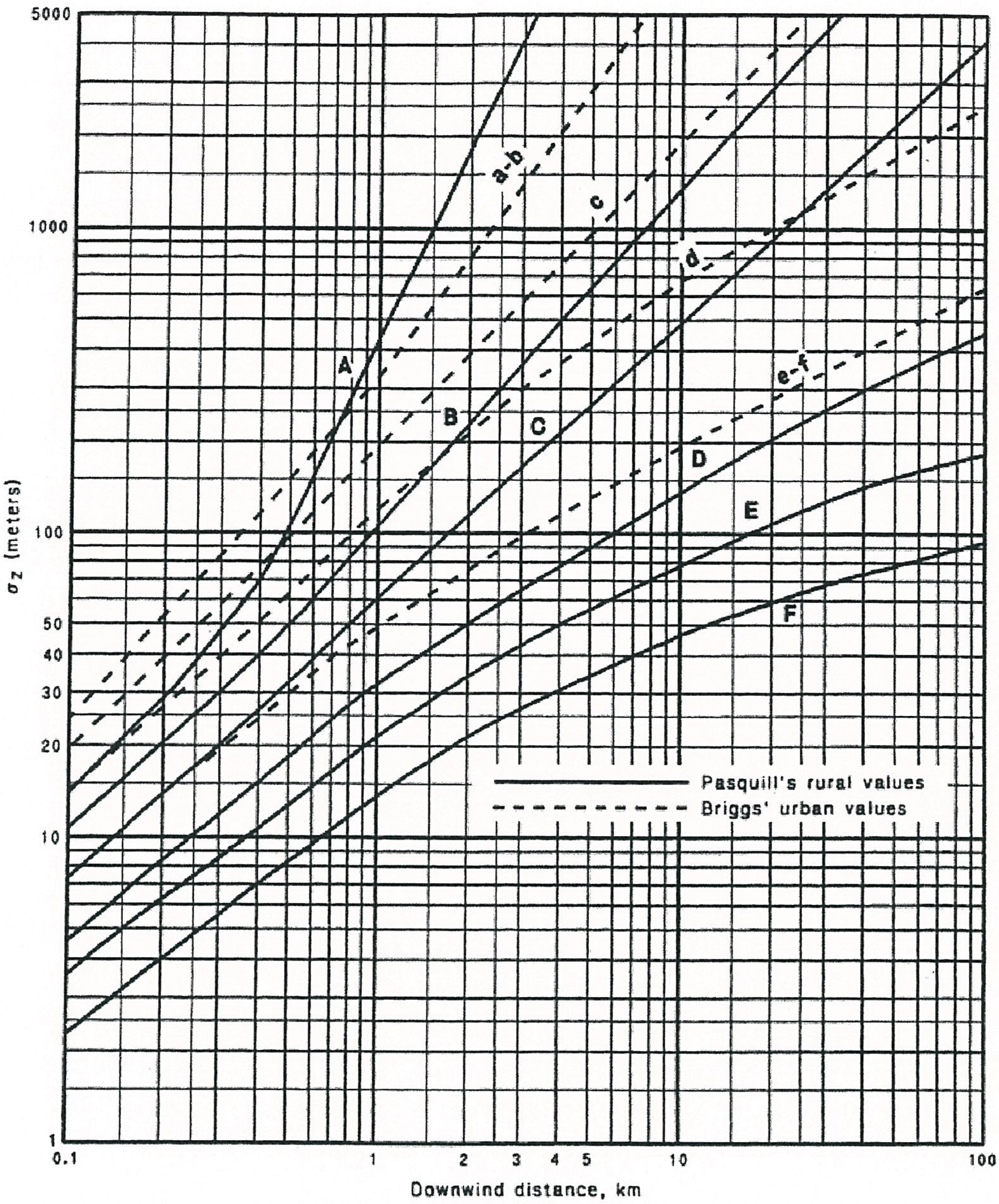
Stability categories and empirical data on dispersion coefficients:

A to F refer progressively from the very unstable to the very stable conditions, with D the neutral.

Wind speed at 10 m high (m/s)	Day			Night	
	Solar intensity				
	Strong	Moderate	Slight	Overcast	Clear
0-2	A	A-B	B	-	-
2-3	A-B	B	C	E	F
3-5	B	B-C	C	D	E
5-6	C	C-D	D	D	D
>6	C	D	D	D	D



Horizontal



5000
1000
100
50
40
30
20
10
1
0.1
1
2
3
4
5
10
100
σz (meters)
Downwind distance, km
A
B
C
D
E
F
a-b
c
d
e-f
Pasquill's rural values
Briggs' urban values

Vertical

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Numerical answers

Q4(b): 4.2×10^{-7} kg/m^3