ENGINEERING TRIPOS PART IIB

Monday 9 May 2011 9 to 10.30

Module 4A10

FLOW INSTABILITY

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*Attachments: 4A10 data sheet (two pages).*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

mpj04

1 (a) With reference to the vortex-sheet flow profile shown in Fig. 1, in which the $x$-velocity is 0 for $z < 0$ and $U$ for $z > 0$, explain the physical mechanism of Kelvin-Helmholtz instability. [20%]

(b) Assume that the velocity potential, $\phi$, on each side of the vortex sheet is

$$\phi(x,t) = \begin{cases} Ux + f(z)e^{st+ikx} & : \quad z > \eta(x,t) \\ g(z)e^{st+ikx} & : \quad z < \eta(x,t) \end{cases}$$

where $\eta(x,t)$ represents a small perturbation of the vortex sheet and has the functional form $\eta(x,t) = \eta_0 \exp(st + ikx)$. The velocity potential on either side of the vortex sheet satisfies Laplace's equation: $\nabla^2\phi = 0$.

(i) Show that the velocity potential has the form:

$$\phi(x,t) = \begin{cases} Ux + A_1 e^{st+ikx-kz} & : \quad z > \eta(x,t) \\ B_1 e^{st+ikx+kz} & : \quad z < \eta(x,t) \end{cases}$$

[20%]

(ii) Using the kinematic constraint that the interface moves with the fluid, show that, for small amplitude disturbances:

$$\begin{aligned} A_1 &= -(s/k + iU)\eta_0 \\ B_1 &= (s/k)\eta_0 \end{aligned}$$

[20%]

(iii) Using the condition that pressure is continuous across the interface, show that the dispersion relation for this problem is given by:

$$s = -\frac{1}{2}kU(i \pm 1)$$

[40%]

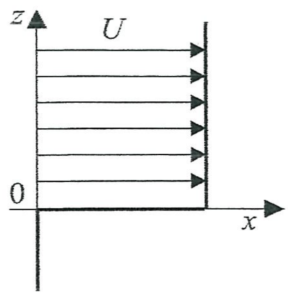



Fig. 1

2 (a) Derive Rayleigh's criterion for the stability of inviscid incompressible flows with circular streamlines. [40%]

(b) The device in Fig. 2 consists of two co-rotating porous cylindrical tubes, which are aligned coaxially. The space between them is filled with fluid, which can move radially inwards or outwards, as well as tangentially. The radial velocity is labelled $U(r)$ and the tangential velocity is labelled $V(r)$. In this question, assume that the flow is uniform in the axial and tangential directions, i.e. that the velocity field $(U,V)$ is a function of $r$ only.

(i) Use continuity to show that $U = \alpha/r$, where $\alpha$ is a constant. [10%]

(ii) If the flow is inviscid, show that $V = \beta/r$, where $\beta$ is a constant. [20%]

(iii) Use Rayleigh's criterion to determine the stability of the flow. [10%]

(c) For a viscous fluid, describe the behaviour as the rotation speed of the inner cylinder is increased. [20%]

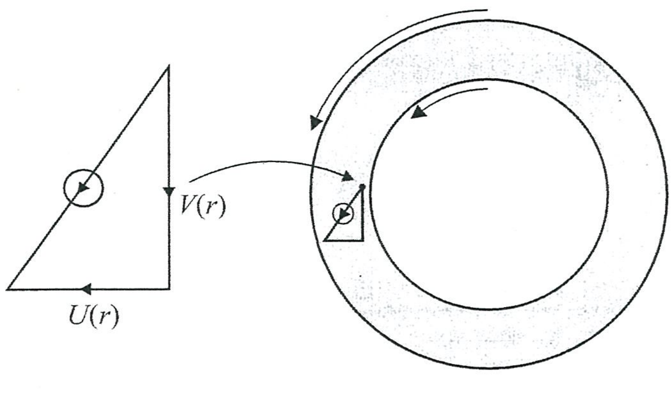



Fig. 2

3 The experiment shown in Fig. 3 consists of an articulated pipe with two segments, both of length $L$. The first segment rotates around the fixed point A against a spring with torsional stiffness $k$. The second segment is attached to the end of the first segment at point B and rotates against a spring with torsional stiffness $k$. The angles that the two segments make with the $y$-axis are $\theta_1$ and $\theta_2$, measured anticlockwise, such that $\theta_1$ as shown in Fig. 3 is negative. Fluid flows through the pipe with speed $U$ and mass flowrate $\dot{m}$. A small force, $F$, is applied in the $x$-direction at the end of the second segment, point C, leading to a small displacement, $X$, in the $x$-direction. The angles $\theta_1$ and $\theta_2$ and the vertical displacement of point C can be assumed to be small. The system can be assumed to be in static equilibrium.

(a) The fluid exerts a centrifugal force on the pipe at point B. Show that this force is equal to $\dot{m}U(\theta_2 + \theta_1)$. [20%]

(b) By considering moments about point B, find a relationship between $F$, $L$, $k$, $\theta_1$ and $\theta_2$. [10%]

(c) By considering moments about point A, find a relationship between $F$, $L$, $k$, $\theta_1$, $\theta_2$, $\dot{m}$ and $U$. [10%]

(d) Find an expression for the displacement, $X$, in terms of $F$. [20%]

(e) Paying particular attention to the relationship between $F$ and $X$, describe how the apparatus behaves as $\dot{m}U$ increases. [20%]

(f) The apparatus is now constrained such that point C is allowed to move vertically, but not horizontally, so that $X = 0$. Describe how the new apparatus behaves as $\dot{m}U$ increases. [20%]

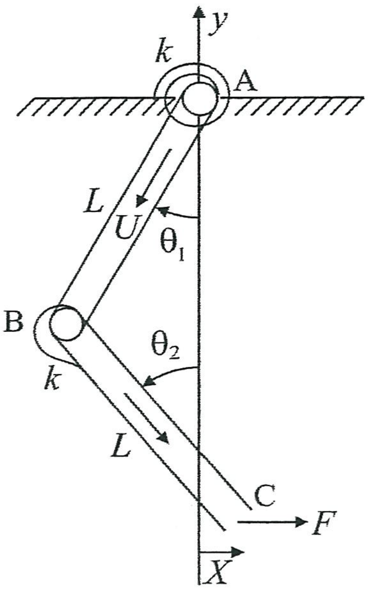



Fig. 3



(TURN OVER

4 A membrane is being dried. As part of this process, shown in Fig. 4, the membrane is held in tension, $T$, between two streams of air moving at speeds $U_1$ and $U_2$ in opposite directions to each other. Considering perturbations on the membrane with amplitude proportional to $\exp\{\mathrm{i}(kx-\omega t)\}$, the dispersion relation is found to be:

$$(U_1+c)^2+(U_2-c)^2-kT=0 \qquad (1)$$

where $c$, the phase speed, is equal to $\omega/k$. It is important to know whether perturbations can develop on the membrane and, if so, whether they grow at the point of inception, which is known as absolutely unstable, or whether they are convected away, which is known as convectively unstable. If the wave with zero group speed has positive growth rate, then the perturbations are absolutely unstable. The group speed, $c_g$, is equal to $\mathrm{d}\omega/\mathrm{d}k$.

(a) Explain what the real and imaginary parts of $\omega$ and $k$ represent physically. [25%]

(b) For real $k$, determine the $k$ above which all disturbances are stable. [25%]

(c) By differentiating (1) with respect to $k$, or otherwise, find the conditions required for the phase speed to be equal to the group speed and comment on your result. [25%]

(d) For $U_1 = U_2 = U$, derive an expression for the group speed. Find the value of $k$ at which it is zero and determine whether or not this system is absolutely unstable. What implications could this have for the membrane during the drying process? [25%]

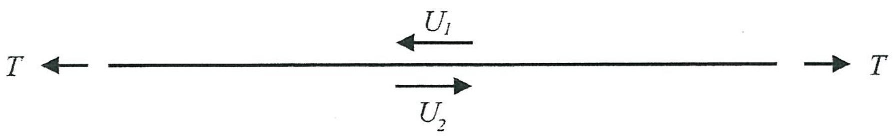



Fig. 4

**END OF PAPER**

# Module 4A10 Data Card

## EQUATIONS OF MOTION

For an incompressible isothermal viscous fluid:

**Continuity** $\nabla \cdot \boldsymbol{u} = 0$

**Navier Stokes** $\rho \dfrac{D\boldsymbol{u}}{Dt} = -\nabla p + \mu \nabla^2 \boldsymbol{u}$

$D/Dt$ denotes the material derivative, $\partial/\partial t + \boldsymbol{u} \cdot \nabla$

## IRROTATIONAL FLOW $\nabla \times \boldsymbol{u} = 0$

**velocity potential** $\phi$,

$$\boldsymbol{u} = \nabla \phi \text{ and } \nabla^2 \phi = 0$$

**Bernoulli's equation**

for inviscid flow $\dfrac{p}{\rho} + \frac{1}{2}|\boldsymbol{u}|^2 + gz + \dfrac{\partial \phi}{\partial t} =$ constant throughout flow field.

## KINEMATIC CONDITION AT A MATERIAL INTERFACE

A surface $z = \eta(x,y,t)$ moves with fluid of velocity $\boldsymbol{u} = (u, v, w)$ if

$$w = \frac{D\eta}{Dt} = \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla \eta \quad \text{on } z = \eta(x,t).$$

For $\eta$ small and $\boldsymbol{u}$ linearly disturbed from $(U,0,0)$

$$w = \frac{\partial \eta}{\partial t} + U \frac{\partial \eta}{\partial x} \quad \text{on } z = 0.$$

## SURFACE TENSION $\sigma$ AT A LIQUID-AIR INTERFACE

**Potential energy**

The potential energy of a surface of area $A$ is $\sigma A$.

**Pressure difference**

The difference in pressure $\Delta p$ across a liquid-air surface with principal radii of curvature $R_1$ and $R_2$ is

$$\Delta p = \sigma \left( \frac{1}{R_1} + \frac{1}{R_2} \right).$$

For a surface which is almost a circular cylinder with axis in the $x$-direction, $r = a + \eta(x, \theta, t)$ ($\eta$ is very small so that $\eta^2$ is negligible)

$$\Delta p = \frac{\sigma}{a} + \sigma \left( -\frac{\eta}{a^2} - \frac{\partial^2 \eta}{\partial x^2} - \frac{1}{a^2} \frac{\partial^2 \eta}{\partial \theta^2} \right),$$

where $\Delta p$ is the difference between the internal and the external surface pressure.

For a surface which is almost plane with $z = \eta(x,t)$ ($\eta$ is very small so that $\eta^2$ is negligible)

$$\Delta p = -\sigma \frac{\partial^2 \eta}{\partial x^2}$$

where $\Delta p$ is the difference between pressure at $z = \eta^+$ and $z = \eta^-$.

## ROTATING FLOW

In steady flows with circular streamlines in which the fluid velocity and pressure are functions of radius $r$ only:

**Rayleigh's criterion**

The flow is unstable / stable to inviscid axisymmetric disturbances if $\Gamma^2$ decreases / increases with $r$.

$\Gamma = 2\pi r V(r)$ is the circulation around a circle of radius $r$.

**Navier Stokes equation simplifies to**

$$0 = \mu \left( \frac{d^2 V}{dr^2} + \frac{1}{r}\frac{dV}{dr} - \frac{V}{r^2} \right)$$

$$-\rho \frac{V^2}{r} = -\frac{dp}{dr}.$$

# Module 4A10 Data Card

## STABILITY OF PARALLEL SHEAR FLOW

**Rayleigh's inflexion point theorem**

A parallel shear flow with profile $U(z)$ is only unstable to inviscid perturbations if

$$\frac{d^2U}{dz^2}=0 \quad \text{for some } z.$$

## CONVECTIVE FLOW

**The Boussinesq approximation** leads to

$$\nabla\cdot \boldsymbol{u}=0$$

$$\frac{D\boldsymbol{u}}{Dt}=-\frac{1}{\rho_0}\nabla p+(1-\alpha(T-T_0))\boldsymbol{g}+\upsilon\nabla^2\boldsymbol{u}$$

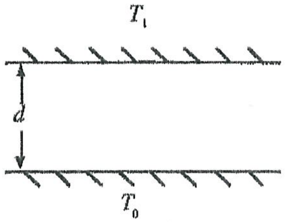
and

$$\frac{DT}{Dt}=\kappa\nabla^2 T$$

**Rayleigh-Bénard convection**

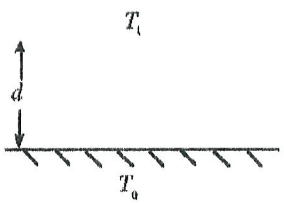
A fluid between two **rigid** plates is unstable when

$$\mathrm{Ra}\ge 1708$$




A liquid with a **free** upper surface is unstable when

$$\frac{\mathrm{Ra}}{\mathrm{Ra}_c}+\frac{\mathrm{Ma}}{\mathrm{Ma}_c}\ge 1$$




where

$$\mathrm{Ra}=\frac{g\alpha(T_0-T_1)d^3}{\upsilon\kappa},\quad \mathrm{Ma}=\frac{\chi(T_0-T_1)d}{\rho\upsilon\kappa}\quad \text{with } \chi=-\frac{d\sigma}{dT}$$

$\mathrm{Ra}_c\approx 670 \qquad \mathrm{Ma}_c\approx 80.$

## USEFUL MATHEMATICAL FORMULA

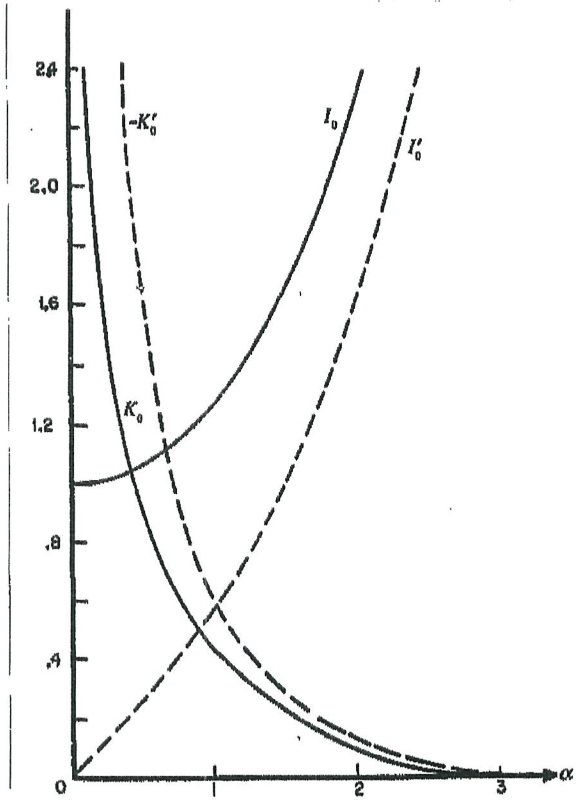
**Modified Bessel equation**

$I_0(kr)$ and $K_0(kr)$ are two independent solutions of

$$\frac{d^2f}{dr^2}+\frac{1}{r}\frac{df}{dr}-k^2f=0.$$

$I_0(kr)$ is finite at $r=0$ and tends to infinity as $r\to\infty$,

$K_0(kr)$ is infinite at $r=0$ and tends to zero as $r\to\infty$.




$I_0(\alpha)$, $K_0(\alpha)$, $I_0'(\alpha)$, $K_0'(\alpha)$
where $'$ denotes a derivative

4A10, 2011, Answers

Q1
(a) –
(b) –

Q2
(a) –
(b) –
(c) –

Q3
(a) –
(b) $FL=k(\theta_2+\theta_1)$
(c) $mUL(\theta_2+\theta_1) = 2FL+k\theta_1$
(d) $X = (FL^2/k)(5 - 2mUL/k)$
(e) –
(f) –

Q4
(a) –
(b) $k > (U_1 + U_2)^2/(2T)$
(c) $2(U_1 - U_2 + 2c)(c_g - c) = kT$, so $c = c_g$ only when $k = 0$ or $T = 0$.
(d) $c_g = c + kT/(4c)$; this is zero when $k = 2U^2/(3T)$; it is absolutely unstable.