ENGINEERING TRIPOS PART IIB

Friday 6 May 2011 9 to 10.30

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4A12 Data Card: (i) Vortex Dynamics (1 page);
(ii) Turbulence (2 pages)

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

PAD03

1 (a) State the Prandtl-Batchelor theorem. [10%]

(b) Consider a steady, two-dimensional flow with closed streamlines. It has a steady temperature field T, thermal diffusivity α, and stream-function ψ.

(i) Starting with the advection-diffusion equation:

$$\mathbf{u} \cdot \nabla T = \alpha \nabla^2 T$$

show that T takes the form $T = T(\psi)$ when $\alpha = 0$.

(ii) Hence show that, when α is small but finite:

$$\mathbf{u} \cdot \nabla T = \alpha \{ \nabla \cdot [T'(\psi) \nabla \psi] + (\text{small correction}) \}$$

where the correction tends to zero as $\alpha \to 0$.

(iii) Use Gauss' theorem to integrate this equation over the area A which is bounded by a streamline C. Confirm that:

$$\alpha \{ T'(\psi) \int \nabla \psi \cdot d\mathbf{S} + (\text{small correction}) \} = 0$$

where $d\mathbf{S} = \mathbf{n} d\ell$ is part of the surface that encircles the area A, $\mathbf{n}$ is a unit normal to the streamline and $d\ell$ is part of that streamline.

(iv) Use the fact that $\int \nabla \psi \cdot d\mathbf{S} = -\oint_C \mathbf{u} \cdot d\mathbf{r}$, where $d\mathbf{r}$ is a short element of the streamline, $|d\mathbf{r}| = d\ell$, to deduce that $T'(\psi) = 0$ in the limit of $\alpha \to 0$. What is the physical interpretation of this result? [50%]

(c) Discuss how the proof of $T'(\psi) = 0$ in part (b) may be adapted to prove the Prandtl-Batchelor theorem. [30%]

(d) What is the physical interpretation of the Prandtl-Batchelor theorem? [10%]

2 (a) State Kelvin's circulation theorem and Helmholtz's two laws of vortex dynamics. [20%]

(b) Consider two adjacent fluid particles, A and B, in an inviscid fluid which are linked by the displacement vector $d\mathbf{r}(t) = \mathbf{x}_B(t) - \mathbf{x}_A(t)$ and which lie on the same vortex line at $t = 0$. Since $d\mathbf{r}$ and $\boldsymbol{\omega}(\mathbf{x}_A)$ are parallel at $t = 0$, we may write $d\mathbf{r}(t = 0) = \lambda\boldsymbol{\omega}(\mathbf{x}_A, t = 0)$, where λ is some scalar. We give λ the property $D\lambda / Dt = 0$, so that it is a constant of the fluid element AB for $t > 0$, and introduce the vector:

$$\mathbf{C}(\mathbf{x}_A, t) = d\mathbf{r}(t) - \lambda\boldsymbol{\omega}(\mathbf{x}_A, t)$$

which is zero at $t = 0$.

(i) Use the fact that a short material line element obeys:

$$\frac{D}{Dt}(d\mathbf{r}) = (d\mathbf{r} \cdot \nabla)\mathbf{u}$$

to show that, in an inviscid fluid:

$$\frac{D\mathbf{C}}{Dt} = \mathbf{C} \cdot \nabla\mathbf{u}$$

(ii) Hence show that, by virtue of the initial conditions, we have $\mathbf{C}(\mathbf{x}_A, t) = 0$ for $t > 0$, and that consequently:

$$\frac{d\mathbf{r}(t)}{|d\mathbf{r}(t = 0)|} = \frac{\boldsymbol{\omega}(\mathbf{x}_A, t)}{|\boldsymbol{\omega}(\mathbf{x}_A, t = 0)|}$$

(iii) Deduce that A and B always lie on the same vortex line and that consequently vortex lines in an inviscid fluid are convected like material lines. [50%]

(c) An alternative proof that vortex lines are frozen into an inviscid fluid can be constructed with the help of Kelvin's theorem. Consider a thin, isolated vortex tube and a closed curve C that encircles the tube at $t = 0$. Use Kelvin's theorem to show that, if C is a material curve always composed of the same fluid particles, then C must encircle the vortex tube for all time. Deduce that vortex lines must move with the fluid, like dye lines. [30%]

3 A hot wire is to be used at the centreline of a turbulent axisymmetric air jet originating from a round nozzle with diameter $d = 10\,\text{mm}$ and with a uniform nozzle velocity $U_J = 50\ \text{ms}^{-1}$. Both the jet fluid and the ambient air are at 1 bar and 300 K. The mean centreline velocity U_0 at a distance x from the nozzle obeys $U_0/U_J = 6.4(x/d)^{-1}$.

(a) By making reasonable guesses for the turbulent integral length-scale and the r.m.s. velocity fluctuations at $x = 50d$, estimate the highest frequency of motion seen by the hot wire at this location. [70%]

(b) For how long should one collect data if we require the mean measured velocity to be within 1% of its true average at this location? [30%]

4 In a dense rain, raindrops of diameter $d = 2$ mm and at a number density of $n = 10^4$ drops / m^3 fall at their terminal velocity $u = 2$ ms^{-1} in air that has no mean flow. Assume that the wakes behind the drops eventually produce stationary homogeneous isotropic turbulence with integral length-scale L and kinetic energy k.

(a) Discuss why we may expect $d < L < n^{-1/3}$. [20%]

(b) Using the extreme values for L from part (a), provide upper and lower bounds for the characteristic turbulent velocity $k^{1/2}$. Find L if $k^{1/2} = u$. [60%]

(c) In a more detailed view of the turbulence in this problem, the wakes behind the drops can be thought of as self-preserving axisymmetric wakes. Given that the half-width $\delta_{1/2}$ of an axisymmetric wake grows as $\delta_{1/2}/d \approx 0.6(x/d)^{1/3}$, where x is the distance behind the drop, provide an alternative estimate for L. [20%]

END OF PAPER

Vortex Dynamics Data Card

Grad, Div and Curl in Cartesian Coordinates

$$\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}, \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}, \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

Integral Theorems

Gauss : $\int (\nabla \cdot A) dV = \oint A \cdot dS$

Stokes : $\int (\nabla \times A) \cdot dS = \oint A \cdot dl$

Vector Identities

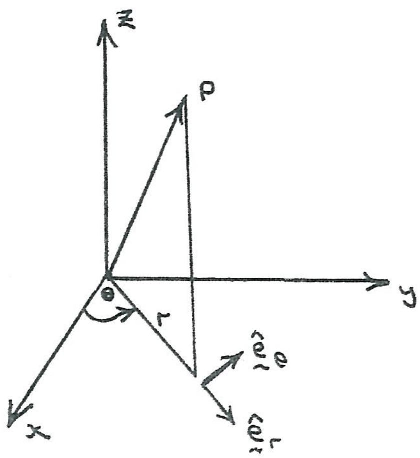
$\nabla(A \cdot B) = (A \cdot \nabla)B + (B.\nabla)A + A \times (\nabla \times B) + B \times (\nabla \times A)$
$\nabla \cdot (fA) = f(\nabla \cdot A) + A \cdot \nabla f$
$\nabla \times (A \times B) = A(\nabla \cdot B) - B(\nabla \cdot A) + (B \cdot \nabla)A - (A \cdot \nabla)B$
$\nabla \times (\nabla \times A) = \nabla(\nabla \cdot A) - \nabla^2 A$
$\nabla \times (\nabla f) = 0$
$\nabla \cdot (\nabla \times A) = 0$

Cylindrical Coordinates (r, θ, z)

$$\nabla f = \left(\frac{\partial f}{\partial r}, \frac{1}{r}\frac{\partial f}{\partial \theta}, \frac{\partial f}{\partial z}\right)$$

$$\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{1}{r}\frac{\partial A_\theta}{\partial \theta} + \frac{\partial A_z}{\partial z}$$

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} \hat{e}_r & r\hat{e}_\theta & \hat{e}_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ A_r & rA_\theta & A_z \end{vmatrix}$$



$$\nabla \times A = \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \frac{\partial A_\theta}{\partial z}, \frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}, \frac{1}{r}\frac{\partial}{\partial r}(rA_\theta) - \frac{1}{r}\frac{\partial A_r}{\partial \theta}\right)$$

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4A12: Turbulence

Data Card

Assume incompressible fluid with constant properties.

Continuity:

$$\frac{\partial \overline{u}_i}{\partial x_i} = 0$$

Mean momentum:

$$\frac{\partial \overline{u}_i}{\partial t} + \overline{u}_j \frac{\partial \overline{u}_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{p}}{\partial x_i} + \nu \partial^2 \overline{u}_i / \partial x_j^2 - \frac{\partial \overline{u_i' u_j'}}{\partial x_j} + \overline{g}_i$$

Mean scalar:

$$\frac{\partial \overline{\phi}}{\partial t} + \overline{u}_i \frac{\partial \overline{\phi}}{\partial x_i} = D \frac{\partial^2 \overline{\phi}}{\partial x_i^2} - \frac{\partial \overline{u_i' \phi'}}{\partial x_i}$$

Turbulent kinetic energy ($k = \overline{u_i' u_i'}/2$):

$$\frac{\partial k}{\partial t} + \overline{u}_j \frac{\partial k}{\partial x_j} = -\frac{1}{\rho}\frac{\partial \overline{u_j' p'}}{\partial x_j} - \frac{1}{2}\frac{\partial \overline{u_j' u_i' u_i'}}{\partial x_j} + \nu \frac{\partial^2 k}{\partial x_j^2} - \overline{u_i' u_j'}\frac{\partial \overline{u}_i}{\partial x_j} - \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} + \overline{g_i' u_i'}$$

Scalar fluctuations ($\sigma^2 = \overline{\phi' \phi'}$):

$$\frac{\partial \sigma^2}{\partial t} + \overline{u}_j \frac{\partial \sigma^2}{\partial x_j} = D \frac{\partial^2 \sigma^2}{\partial x_j^2} - 2\overline{\phi' u_j' \frac{\partial \phi'}{\partial x_j}} - 2\overline{\phi' u_j'}\frac{\partial \overline{\phi}}{\partial x_j} - 2D \overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2}$$

Energy dissipation:

$$\varepsilon = \nu \overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \approx \frac{u^3}{L_{turb}}$$

Scalar dissipation:

$$2\overline{N} = 2D\overline{\left(\frac{\partial \phi'}{\partial x_j}\right)^2} \approx 2\frac{\varepsilon}{k}\sigma^2$$

Scaling rule for shear flow, flow dominant in direction x_1:

$$\frac{u}{L_{turb}} \sim \frac{\partial \overline{u}_1}{\partial x_2}$$

Kolmogorov scales:

$$\begin{aligned}
\eta_K &= \left(\nu^3/\varepsilon\right)^{1/4} \\
\tau_K &= \left(\nu/\varepsilon\right)^{1/2} \\
v_K &= \left(\nu\varepsilon\right)^{1/4}
\end{aligned}$$

Taylor microscale:

$$\varepsilon = 15\nu\frac{u^2}{\lambda^2}$$

Eddy viscosity (general):

$$\begin{aligned}
\overline{u_i'u_j'} &= -\nu_{turb}\left(\frac{\partial \overline{u}_i}{\partial x_j} + \frac{\partial \overline{u}_j}{\partial x_i}\right) + \frac{2}{3}k\delta_{ij} \\
\overline{u_j'\phi'} &= -D_{turb}\frac{\partial \overline{\phi}}{\partial x_j}
\end{aligned}$$

Eddy viscosity (for simple shear):

$$\overline{u_1'u_2'} = -\nu_{turb}\frac{\partial \overline{u}_1}{\partial x_2}$$

ENGINEERING TRIPOS PART IIB 2011

MODULE 4A12: TURBULENCE AND VORTEX DYNAMICS.

NUMERICAL ANSWERS

Q1. –
Q2. –
Q3. (a) 80kHz, (b) 4.9 s
Q4. (a) - , (b) L = 10.7 m, (c) L = 3.4 mm