ENGINEERING TRIPOS PART IIB

Wednesday 11 May 2011 9 to 10.30

Module 4C2

DESIGNING WITH COMPOSITES

Answer not more than **three** *questions.*

All questions carry the same number of marks.

The **approximate** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4C2 datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Why are composite laminates often designed to be balanced and symmetric? [30%]

(b) An epoxy-glass composite with elastic constants $E_1 = 30$ GPa, $E_2 = 6$ GPa, $G_{12} = 2$ GPa, and $\nu_{12} = 0.25$ is subject to uniaxial tension. Calculate the loading angle for which $\left|\bar{S}_{16}\right|$ is:

(i) zero; [25%]

(ii) a maximum. [25%]

(c) Hence sketch the variation of $\left|\bar{S}_{16}\right|$ with θ for $0 \le \theta \le 90°$. Comment on the physical significance of $\left|\bar{S}_{16}\right|$ in designing laminates. [20%]

2 Discuss the design and manufacture of a composite spoiler (a control surface on the top of an aircraft wing), as sketched in Fig. 1. Include the following issues in your discussion (but do not limit yourself to just these points):

- choice of material;
- process route, including a brief description of your suggested method;
- structural design and any feature details;
- testing and certification. [100%]

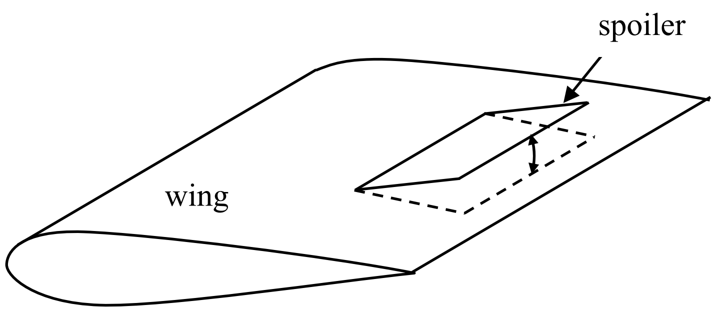


Fig. 1.

3 Figure 2 illustrates a model for tow pull-out across a laminate break in a woven composite material under the action of a uniaxial tensile stress σ_0. The figure shows a unit cell of square cross-section with side length a containing a square-section reinforcing tow of side length b. The tow bridges the laminate break and has itself broken a distance L from the laminate break. Pull-out of the tow from the laminate is resisted by a frictional stress τ.

(a) Suggest why the tow might have broken at a different location from the main laminate break. [10%]

(b) Derive an expression for the axial stress in the tow as a function of the distance x from the tow break, for $0 \le x \le L$. Show that the remote stress σ_0 needed to pull out the tow is given by

$$\sigma_0 = \frac{4\tau bL}{a^2}$$ [30%]

(c) Hence find an expression for the tow pull-out energy per unit cross section of laminate, in terms of τ, b, a and L. For a CFRP composite, with the data given below, would this pull-out mechanism give a useful contribution to the fracture energy? **Data:** $a = 5$ mm, $b = 1$ mm, $L = 20$ mm, $\tau = 50$ MPa. [30%]

(d) Explain qualitatively how you would expect the tow stress to vary with position to the left of the tow break. Derive an expression for the variation in this stress. The laminate and tow moduli in the direction of the applied stress are E_L and E_T, respectively. [30%]

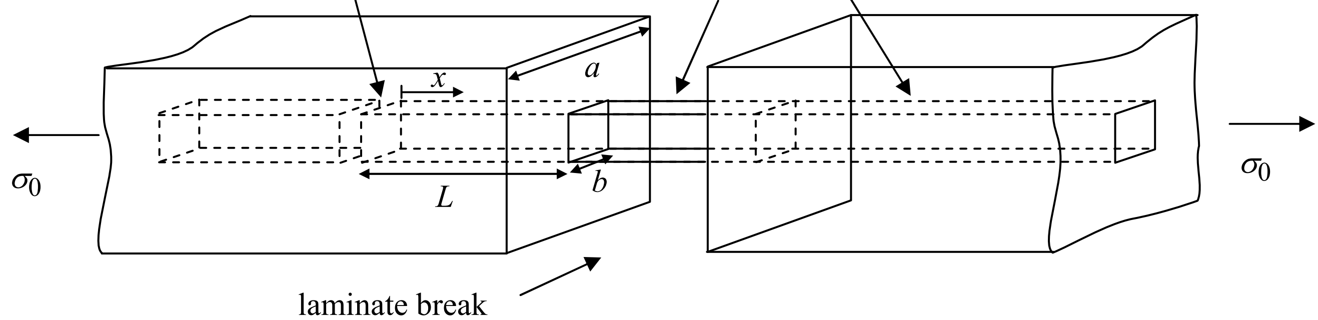


Fig. 2.

4 Figure 3 shows a laminate made of three identical woven GFRP plies, each of thickness t. The outside plies have fibres orientated at 0° and 90° to the applied tensile line load N, while the mid-ply is orientated at ±45°. The $[Q]$ matrix for the 0/90° ply is given by

$$[Q] = E\begin{bmatrix} 1 & 0.1 & 0 \\ 0.1 & 1 & 0 \\ 0 & 0 & 0.2 \end{bmatrix}$$

where E is a material constant.

(a) Obtain the $[\overline{Q}]$ matrix for the ±45° ply in axes aligned with the loading direction. [20%]

(b) Explain why the laminate behaves in a balanced and symmetric manner. [10%]

(c) Show that the $[A]$ matrix for the laminate is given by

$$[A] = Et\begin{bmatrix} 2.75 & 0.55 & 0 \\ 0.55 & 2.75 & 0 \\ 0 & 0 & 0.85 \end{bmatrix}$$ [15%]

(d) Failure is characterised by the Tsai-Hill failure criterion with a shear strength s_{LT}, and tensile and compressive strengths s^+ and s^-, respectively, for loading along the fibre directions of the ply. Derive an expression for the line load N which would cause tensile failure in the 0/90° plies, in terms of t, E and the strength parameters. [55%]

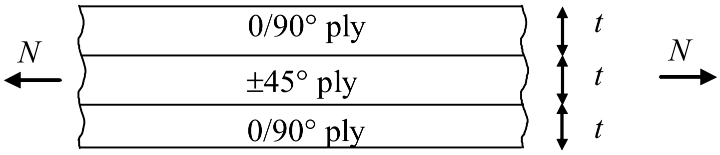


Fig. 3.

END OF PAPER

ENGINEERING TRIPOS PART II B

Module 4C2 – Designing with Composites

DATA SHEET

The in-plane compliance matrix [S] for a transversely isotropic lamina is defined by

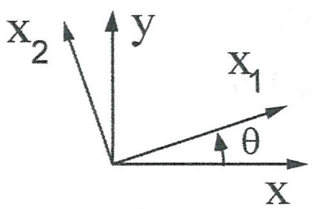
$$\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[S]\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix} \qquad \text{where } [S]=\begin{bmatrix} 1/E_1 & -\nu_{21}/E_2 & 0\\ -\nu_{12}/E_1 & 1/E_2 & 0\\ 0 & 0 & 1/G_{12}\end{bmatrix}$$

[S] is symmetric, giving $\nu_{12}/E_1=\nu_{21}/E_2$. The compliance relation can be inverted to give

$$\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=\begin{bmatrix} Q_{11} & Q_{12} & 0\\ Q_{12} & Q_{22} & 0\\ 0 & 0 & Q_{66}\end{bmatrix}\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix} \qquad \begin{aligned} \text{where } Q_{11} &= E_1/(1-\nu_{12}\nu_{21})\\ Q_{22} &= E_2/(1-\nu_{12}\nu_{21})\\ Q_{12} &= \nu_{12}E_2/(1-\nu_{12}\nu_{21})\\ Q_{66} &= G_{12}\end{aligned}$$

Rotation of co-ordinates

Assume the principal material directions (x_1, x_2) are rotated anti-clockwise by an angle θ, with respect to the (x,y) axes.



Then, $\begin{pmatrix}\sigma_1\\ \sigma_2\\ \sigma_{12}\end{pmatrix}=[T]\begin{pmatrix}\sigma_x\\ \sigma_y\\ \sigma_{xy}\end{pmatrix}$ and $\begin{pmatrix}\varepsilon_1\\ \varepsilon_2\\ \gamma_{12}\end{pmatrix}=[T]^{-T}\begin{pmatrix}\varepsilon_x\\ \varepsilon_y\\ \gamma_{xy}\end{pmatrix}$

where $$[T]=\begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta\\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta-\sin^2\theta\end{bmatrix}$$

and $$[T]^{-T}=\begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta\\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta\\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & (\cos^2\theta-\sin^2\theta)\end{bmatrix}$$

The stiffness matrix $[Q]$ transforms in a related manner to the matrix $[\overline{Q}]$ when the axes are rotated from (x_1, x_2) to (x, y)

$$[\overline{Q}] = [T]^{-1}[Q][T]^{-T}$$

In component form,

$$[\overline{Q}] = \begin{bmatrix} \overline{Q}_{11} & \overline{Q}_{12} & \overline{Q}_{16} \\ \overline{Q}_{12} & \overline{Q}_{22} & \overline{Q}_{26} \\ \overline{Q}_{16} & \overline{Q}_{26} & \overline{Q}_{66} \end{bmatrix} \quad \text{where} \quad \begin{aligned} \overline{Q}_{11} &= Q_{11}c^4 + Q_{22}s^4 + 2(Q_{12} + 2Q_{66})s^2c^2 \\ \overline{Q}_{12} &= (Q_{11} + Q_{22} - 4Q_{66})s^2c^2 + Q_{12}(c^4 + s^4) \\ \overline{Q}_{22} &= Q_{11}s^4 + Q_{22}c^4 + 2(Q_{12} + 2Q_{66})s^2c^2 \\ \overline{Q}_{16} &= (Q_{11} - Q_{12} - 2Q_{66})sc^3 - (Q_{22} - Q_{12} - 2Q_{66})s^3c \\ \overline{Q}_{26} &= (Q_{11} - Q_{12} - 2Q_{66})s^3c - (Q_{22} - Q_{12} - 2Q_{66})sc^3 \\ \overline{Q}_{66} &= (Q_{11} + Q_{22} - 2Q_{12} - 2Q_{66})s^2c^2 + Q_{66}(s^4 + c^4) \end{aligned}$$

with $c = \cos\theta, \quad s = \sin\theta$

The compliance matrix $[S] \equiv [Q]^{-1}$ transforms to $[\overline{S}] \equiv [\overline{Q}]^{-1}$ under a rotation of co-ordinates by θ from (x_1, x_2) to (x, y), as

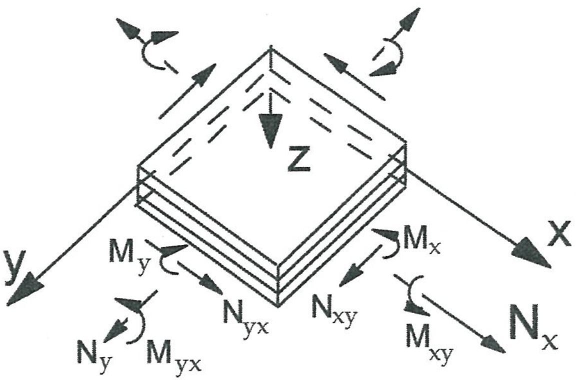
$$[\overline{S}] = [T]^T [S][T]$$

and in component form,

$$\begin{aligned} \overline{S}_{11} &= S_{11}c^4 + S_{22}s^4 + (2S_{12} + S_{66})s^2c^2 \\ \overline{S}_{12} &= S_{12}(c^4 + s^4) + (S_{11} + S_{22} - S_{66})s^2c^2 \\ \overline{S}_{22} &= S_{11}s^4 + S_{22}c^4 + (2S_{12} + S_{66})s^2c^2 \\ \overline{S}_{16} &= (2S_{11} - 2S_{12} - S_{66})sc^3 - (2S_{22} - 2S_{12} - S_{66})s^3c \\ \overline{S}_{26} &= (2S_{11} - 2S_{12} - S_{66})s^3c - (2S_{22} - 2S_{12} - S_{66})sc^3 \\ \overline{S}_{66} &= (4S_{11} + 4S_{22} - 8S_{12} - 2S_{66})s^2c^2 + S_{66}(c^4 + s^4) \end{aligned}$$

with $c = \cos\theta, \quad s = \sin\theta$

Laminate Plate Theory



Consider a plate subjected to stretching of the mid-plane by $\left(\varepsilon_x^o, \varepsilon_y^o, \varepsilon_{xy}^o\right)^T$ and to a curvature $\left(\kappa_x, \kappa_y, \kappa_{xy}\right)^T$. The stress resultants $\left(N_x, N_y, N_{xy}\right)^T$ and bending moment per unit length $\left(M_x, M_y, M_{xy}\right)^T$ are given by

$$\begin{pmatrix} N \\ \cdots \\ M \end{pmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & . & \cdots \\ B & \vdots & D \end{bmatrix} \begin{pmatrix} \varepsilon^o \\ \cdots \\ \kappa \end{pmatrix}$$

In component form, we have,

$$\begin{pmatrix} N_x \\ N_y \\ N_{xy} \\ M_x \\ M_y \\ M_{xy} \end{pmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} & B_{11} & B_{12} & B_{16} \\ A_{12} & A_{22} & A_{26} & B_{12} & B_{22} & B_{26} \\ A_{16} & A_{26} & A_{66} & B_{16} & B_{26} & B_{66} \\ B_{11} & B_{12} & B_{16} & D_{11} & D_{12} & D_{16} \\ B_{12} & B_{22} & B_{26} & D_{12} & D_{22} & D_{26} \\ B_{16} & B_{26} & B_{66} & D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{pmatrix} \varepsilon_x^o \\ \varepsilon_y^o \\ \gamma_{xy}^o \\ \kappa_x \\ \kappa_y \\ \kappa_{xy} \end{pmatrix}$$

where the laminate extensional stiffness, A_{ij}, is given by:

$$A_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k dz = \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k - z_{k-1}\right)$$

the laminate coupling stiffnesses is given by

$$B_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z dz = \frac{1}{2} \sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^2 - z_{k-1}^2\right)$$

and the laminate bending stiffness are given by:

$$D_{ij} = \int_{-t/2}^{t/2} \left(\bar{Q}_{ij}\right)_k z^2 dz = \frac{1}{3}\sum_{k=1}^{n} \left(\bar{Q}_{ij}\right)_k \left(z_k^3 - z_{k-1}^3\right)$$

with the subscripts $i, j = 1, 2$ or 6.

Here,

$n =$ number of laminae

$t =$ laminate thickness

$z_{k\text{-}1} =$ distance from middle surface to the top surface of the *k-th* lamina

$z_k =$ distance from middle surface to the bottom surface of the *k-th* lamina

Quadratic failure criteria.

For plane stress with $\sigma_3 = 0$, failure is predicted when

Tsai-Hill: $\dfrac{\sigma_1^2}{s_L^2} - \dfrac{\sigma_1\sigma_2}{s_L^2} + \dfrac{\sigma_2^2}{s_T^2} + \dfrac{\tau_{12}^2}{s_{LT}^2} \geq 1$

Tsai-Wu: $F_{11}\sigma_1^2 + F_{22}\sigma_2^2 + F_{66}\tau_{12}^2 + F_1\sigma_1 + F_2\sigma_2 + 2F_{12}\sigma_1\sigma_2 \geq 1$

where $F_{11} = \dfrac{1}{s_L^+ s_L^-}$, $F_{22} = \dfrac{1}{s_T^+ s_T^-}$, $F_1 = \dfrac{1}{s_L^+} - \dfrac{1}{s_L^-}$, $F_2 = \dfrac{1}{s_T^+} - \dfrac{1}{s_T^-}$, $F_{66} = \dfrac{1}{s_{LT}^2}$

F_{12} should ideally be optimised using appropriate strength data. In the absence of such data, a default value which should be used is

$$F_{12} = -\frac{(F_{11}F_{22})^{1/2}}{2}$$

Fracture mechanics

Consider an orthotropic solid with principal material directions x_1 and x_2. Define two effective elastic moduli E'_A and E'_B as

$$\frac{1}{E'_A} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{22}}{S_{11}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

$$\frac{1}{E'_B} = \left(\frac{S_{11}S_{22}}{2}\right)^{1/2}\left(\left(\frac{S_{11}}{S_{22}}\right)^{1/2}\left(1+\frac{2S_{12}+S_{66}}{2\sqrt{S_{11}S_{22}}}\right)\right)^{1/2}$$

where S_{11} etc. are the compliances.

Then G and K are related for plane stress conditions by:

crack running in x_1 direction: $G_I E'_A = K_I^2; G_{II} E'_B = K_{II}^2$

crack running in x_2 direction: $G_I E'_B = K_I^2; G_{II} E'_A = K_{II}^2$.

For mixed mode problems, the total strain energy release rate G is given by

$G = G_I + G_{II}$

Approximate design data

	Steel	Aluminium	CFRP	GFRP	Kevlar
Cost C (£/kg)	1	2	100	5	25
E_1 (GPa)	210	70	140	45	80
G (GPa)	80	26	≈35	≈11	≈20
ρ (kg/m^3)	7800	2700	1500	1900	1400
e^+ (%)	0.1-0.8	0.1-0.8	0.4	0.3	0.5
e^- (%)	0.1-0.8	0.1-0.8	0.5	0.7	0.1
e_{LT} (%)	0.15-1	0.15-1	0.5	0.5	0.3

Table 1. Material data for preliminary or conceptual design. Costs are very approximate.

	Aluminium	Carbon/epoxy (AS/3501)	Kevlar/epoxy (Kevlar 49/934)	E-glass/epoxy (Scotchply/1002)
Cost (£/kg)	2	100	25	5
Density (kg/m^3)	2700	1500	1400	1900
E_1 (GPa)	70	138	76	39
E_2 (GPa)	70	9.0	5.5	8.3
ν_{12}	0.33	0.3	0.34	0.26
G_{12} (GPa)	26	6.9	2.3	4.1
s_L^+ (MPa)	300 (yield)	1448	1379	1103
s_L^- (MPa)	300	1172	276	621
s_T^+ (MPa)	300	48.3	27.6	27.6
s_T^- (MPa)	300	248	64.8	138
s_{LT} (MPa)	300	62.1	60.0	82.7

Table 2. Material data for detailed design calculations. Costs are very approximate.

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October 2008

Engineering Tripos Part IIB: Module 4C2
Designing with Composites

Numerical answers - 2010/11

1. (b) (i) 0, 90, 59°
    (ii) local maxima at 27 or 73°, larger at 27°

3. (c) 1600 kJ/m^2