ENGINEERING TRIPOS PART IIB

Wednesday 11 May 2011 2.30 to 4.00

Module 4C6

ADVANCED LINEAR VIBRATIONS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4C6 Advanced Linear Vibration data sheet (10 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 An impulse hammer of mass 0.005 kg is to be used to conduct a modal test on the back of a guitar whose mass is around 0.1 kg. A theoretical model suggests that the nodal lines of the first six mode shapes and the corresponding natural frequencies are those shown in Fig. 1.

(a) Determine the approximate duration of the impulse necessary to excite the first six modes satisfactorily without exciting higher modes. [10%]

(b) What value of the stiffness of the hammer tip will give a suitable pulse? [10%]

(c) The force transducer has a sensitivity of around 1 pCN^{-1} and the hammer strikes the guitar at a speed of around 1 ms^{-1}. Design a simple charge amplifier to produce an output signal suitable for a data logger with an input range of ± 5 V. [15%]

(d) Select a suitable sampling rate for the data logger so that the use of an anti-aliasing filter is not required. [5%]

(e) Sketch the form of the magnitude of the transfer function that might be measured if an accelerometer of negligible mass is fixed on the guitar at point B (shown in Fig. 1) and the impulse is applied at point A. [30%]

(f) If a mass of 0.02 kg were to be glued to the guitar at the point C describe with sketches how you might expect the mode shapes, modal frequencies and transfer function to change. [30%]

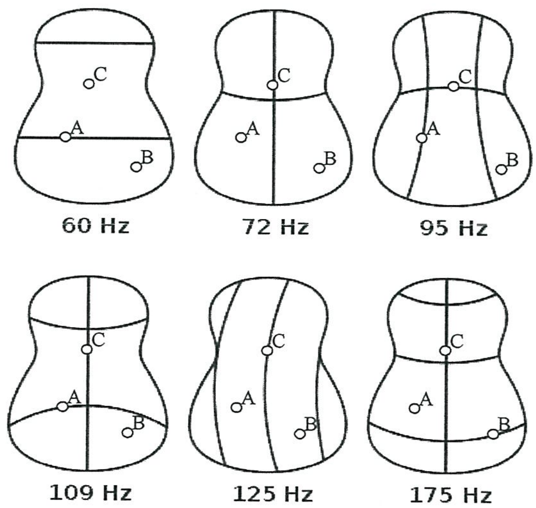


Fig. 1

2 (a) The damping properties of materials can be measured using three approaches: transient decay of vibration; behaviour near peaks of vibration transfer functions in the frequency domain; the response to forced vibration. Outline the advantages and disadvantages of each method. With reference to information given in the Data Sheet, explain which method you would choose to measure the damping of

(i) glass;

(ii) PMMA;

(iii) butyl rubber. [50%]

(b) A beam made of low-damping tool steel is to be given a coating on one side only, to increase its vibration damping to reduce fatigue damage. The coating is to be much thinner than the beam itself. Using information from the Data Sheet on free-layer damping, obtain an approximate expression for the effective damping factor η of the coated beam. Do not assume that the coating material necessarily has light damping, but ignore the damping of the tool steel. [30%]

(c) Explain how the chart of material data for Young's modulus and damping, given in the Data Sheet, can be used to select the best material for the coating, in the sense of maximising η. Would you choose a metal or a polymer for this task? [20%]

3 (a) Explain briefly the mechanism by which a Helmholtz resonator works. Using the notation and information from the Data Sheet, give expressions for the mass and stiffness of the effective mass-spring oscillator which represents the Helmholtz resonance. [25%]

(b) Two Helmholtz resonators have the same neck details but different volumes, and they are connected together by a narrow tube which allows some air flow governed by viscous force. The system can be approximately represented by the mass-spring-dashpot arrangement shown in Fig. 2, with two equal masses m, two springs of stiffness k_1 and k_2, and a dashpot of strength c. The dashpot strength may be assumed to be very small. Write down the mass, stiffness and damping matrices, and hence obtain the governing equations of motion for the system in first-order form. [35%]

(c) Assuming a form

$$\mathbf{y} = \begin{pmatrix} 1 \\ \alpha \\ \lambda \\ \lambda\alpha \end{pmatrix}$$

for one mode of the coupled system, where λ is the relevant eigenvalue of the matrix A given in the Data Sheet, show that the following equations must be approximately satisfied:

$$-k_1 - \lambda c \approx m\lambda^2$$
$$-\alpha k_2 + \lambda c \approx m\alpha\lambda^2$$

if it is assumed that $|\alpha| << 1$. Hence obtain an approximate expression, accurate to first order in c, for the frequency and damping factor of this mode, and show that the two masses move with a phase difference of approximately 90°. [40%]

(CONT.

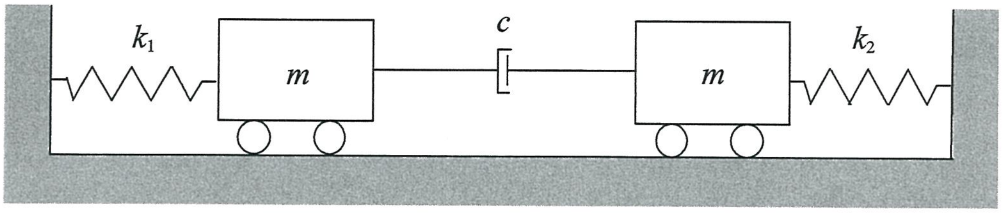


Fig. 2

4 (a) A circular membrane with radius a, tension per unit length T and mass per unit area m is fixed around its perimeter. Using information from the Data Sheet, find expressions for the first 6 natural frequencies. [25%]

(b) The centre point of the membrane is now constrained so it cannot move. Which of the natural frequencies change and which remain the same? Considering the axisymmetric modes alone, what does the interlacing theorem say about the possible constrained natural frequencies? Sketch a guess at the first axisymmetric mode shape. [35%]

(c) To investigate more carefully the first natural frequency of the constrained membrane, suppose that the membrane is fixed not just at a single point, but around an inner circle of radius $b \ll a$. The axisymmetric solutions consist of a linear combination of the Bessel function $J_0(kr)$ and the second solution to Bessel's equation, called $Y_0(kr)$. Both functions are plotted in Fig. 3: $J_0(kr)$ in the solid line and $Y_0(kr)$ in the dashed line. Note that $Y_0(kr) \to -\infty$ as $r \to 0$. Explain briefly how you would construct a solution satisfying the fixed boundary conditions at $r = a$ and $r = b$. Now consider what happens as $b \to 0$, and hence find the lowest frequency of the point-constrained membrane. [40%]

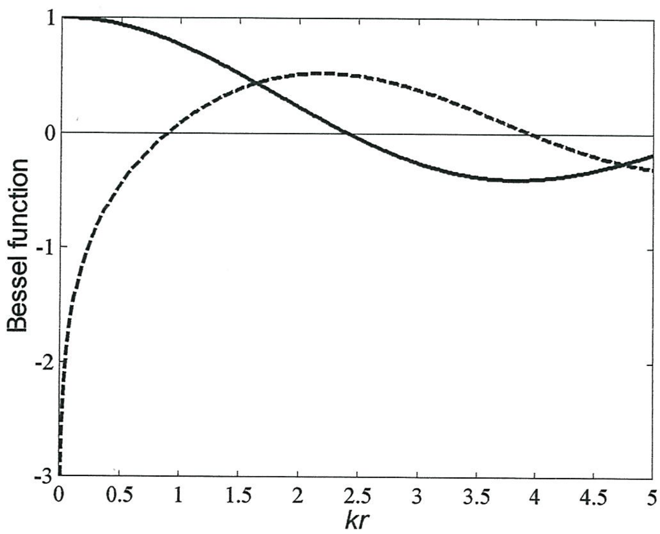


Fig. 3

END OF PAPER

Part IIB Data sheet

Module 4C6 Advanced linear vibration

VIBRATION MODES AND RESPONSE

Discrete systems

1. The forced vibration of an N-degree-of-freedom system with mass matrix M and stiffness matrix K (both symmetric and positive definite) is

$$M\,\ddot{\underline{y}} + K\,\underline{y} = \underline{f}$$

where $\underline{y}$ is the vector of generalised displacements and $\underline{f}$ is the vector of generalised forces.

2. Kinetic energy

$$T = \frac{1}{2}\dot{\underline{y}}^t M \dot{\underline{y}}$$

Potential energy

$$V = \frac{1}{2}\underline{y}^t K \underline{y}$$

3. The natural frequencies ω_n and corresponding mode shape vectors $\underline{u}^{(n)}$ satisfy

$$K\underline{u}^{(n)} = \omega_n{}^2 M \underline{u}^{(n)} \,.$$

4. Orthogonality and normalisation

$$\underline{u}^{(j)t} M \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

$$\underline{u}^{(j)t} K \underline{u}^{(k)} = \begin{cases} 0, & j \neq k \\ \omega_n^2, & j = k \end{cases}$$

Continuous systems

The forced vibration of a continuous system is determined by solving a partial differential equation: see p. 4 for examples.

$$T = \frac{1}{2}\int \dot{u}^2\, dm$$

where the integral is with respect to mass (similar to moments and products of inertia).

See p. 4 for examples.

The natural frequencies ω_n and mode shapes $u_n(x)$ are found by solving the appropriate differential equation (see p. 4) and boundary conditions, assuming harmonic time dependence.

$$\int u_j(x)\, u_k(x)\, dm = \begin{cases} 0, & j \neq k \\ 1, & j = k \end{cases}$$

5. General response

The general response of the system can be written as a sum of modal responses

$$\underline{y}(t) = \sum_{j=1}^{N} q_j(t)\, \underline{u}^{(j)} = U\underline{q}(t)$$

where U is a matrix whose N columns are the normalised eigenvectors $\underline{u}^{(j)}$ and q_j can be thought of as the "quantity" of the jth mode.

The general response of the system can be written as a sum of modal responses

$$w(x,t) = \sum_{j} q_j(t)\, u_j(x)$$

where $w(x,t)$ is the displacement and q_j can be thought of as the "quantity" of the jth mode.

6. Modal coordinates $\underline{q}$ satisfy

$$\ddot{\underline{q}} + \left[\mathrm{diag}(\omega_j^2)\right]\underline{q} = \underline{Q}$$

where $\underline{y} = U\underline{q}$ and the modal force vector

$$\underline{Q} = U^t \underline{f}\ .$$

Each modal amplitude $q_j(t)$ satisfies

$$\ddot{q}_j + \omega_j^2\, q_j = Q_j$$

where $Q_j = \int f(x,t)\, u_j(x)\, dm$ and $f(x,t)$ is the external applied force distribution.

7. Frequency response function

For input generalised force f_j at frequency ω and measured generalised displacement y_k the transfer function is

$$H(j,k,\omega) = \frac{y_k}{f_j} = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(j,k,\omega) = \frac{y_k}{f_j} \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

For force F at frequency ω applied at point x, and displacement w measured at point y, the transfer function is

$$H(x,y,\omega) = \frac{w}{F} = \sum_{n} \frac{u_n(x)\, u_n(y)}{\omega_n^2 - \omega^2}$$

(with no damping), or

$$H(x,y,\omega) = \frac{w}{F} \approx \sum_{n} \frac{u_n(x)\, u_n(y)}{\omega_n^2 + 2i\omega\omega_n\zeta_n - \omega^2}$$

(with small damping) where the damping factor ζ_n is as in the Mechanics Data Book for one-degree-of-freedom systems.

8. Pattern of antiresonances

For a system with well-separated resonances (low modal overlap), if the factor $u_j^{(n)} u_k^{(n)}$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

For a system with low modal overlap, if the factor $u_n(x)\, u_n(y)$ has the same sign for two adjacent resonances then the transfer function will have an antiresonance between the two peaks. If it has opposite sign, there will be no antiresonance.

9. Impulse response

For a unit impulsive generalised force $f_j = \delta(t)$ the measured response y_k is given by

$$g(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \ge 0$ (with small damping).

For a unit impulse applied at $t = 0$ at point x, the response at point y is

$$g(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t$$

for $t \ge 0$ (with no damping), or

$$g(x,y,t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n} \sin\omega_n t \; e^{-\omega_n \zeta_n t}$$

for $t \ge 0$ (with small damping).

10. Step response

For a unit step generalised force $f_j = \begin{cases} 0 & t < 0 \\ 1 & t \ge 0 \end{cases}$ the measured response y_k is given by

$$h(j,k,t) = y_k(t) = \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(j,k,t) \approx \sum_{n=1}^{N} \frac{u_j^{(n)} u_k^{(n)}}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

For a unit step force applied at $t = 0$ at point x, the response at point y is

$$h(x,y,t) = \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t\right]$$

for $t \ge 0$ (with no damping), or

$$h(t) \approx \sum_n \frac{u_n(x)\, u_n(y)}{\omega_n^2} \left[1 - \cos\omega_n t \; e^{-\omega_n \zeta_n t}\right]$$

for $t \ge 0$ (with small damping).

Rayleigh's principle for small vibrations

The "Rayleigh quotient" for a discrete system is $\dfrac{V}{\tilde{T}} = \dfrac{\underline{y}^t K \underline{y}}{\underline{y}^t M \underline{y}}$ where $\underline{y}$ is the vector of generalised coordinates, M is the mass matrix and K is the stiffness matrix. The equivalent quantity for a continuous system is defined using the energy expressions on p. 4.

If this quantity is evaluated with any vector $\underline{y}$, the result will be

(1) $\ge$ the smallest squared frequency;

(2) $\le$ the largest squared frequency;

(3) a good approximation to ω_k^2 if $\underline{y}$ is an approximation to $\underline{u}^{(k)}$.

(Formally, $\dfrac{V}{\tilde{T}}$ is *stationary* near each mode.)

GOVERNING EQUATIONS FOR CONTINUOUS SYSTEMS

Transverse vibration of a stretched string

Tension P, mass per unit length m, transverse displacement $w(x,t)$, applied lateral force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$m\dfrac{\partial^2 w}{\partial t^2} - P\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}P\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}m\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Torsional vibration of a circular shaft

Shear modulus G, density ρ, external radius a, internal radius b if shaft is hollow, angular displacement $\theta(x,t)$, applied torque $f(x,t)$ per unit length.
Polar moment of area is $J = (\pi/2)(a^4 - b^4)$.

Equation of motion	Potential energy	Kinetic energy
$\rho J\dfrac{\partial^2 \theta}{\partial t^2} - GJ\dfrac{\partial^2 \theta}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}GJ\int\left(\dfrac{\partial \theta}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho J\int\left(\dfrac{\partial \theta}{\partial t}\right)^2 dx$

Axial vibration of a rod or column

Young's modulus E, density ρ, cross-sectional area A, axial displacement $w(x,t)$, applied axial force $f(x,t)$ per unit length.

Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} - EA\dfrac{\partial^2 w}{\partial x^2} = f(x,t)$	$V = \dfrac{1}{2}EA\int\left(\dfrac{\partial w}{\partial x}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Bending vibration of an Euler beam

Young's modulus E, density ρ, cross-sectional area A, second moment of area of cross-section I, transverse displacement $w(x,t)$, applied transverse force $f(x,t)$ per unit length.

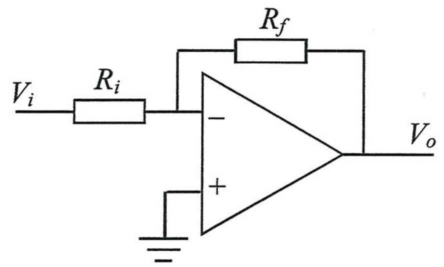
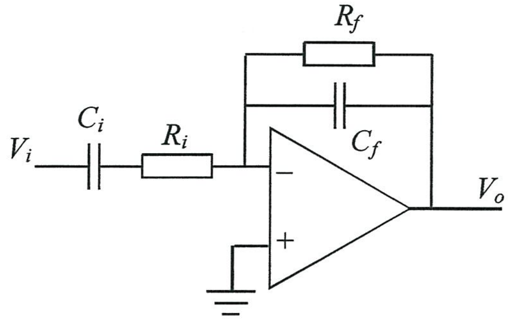
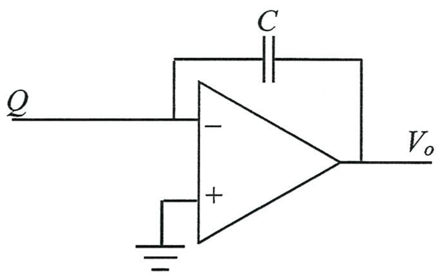
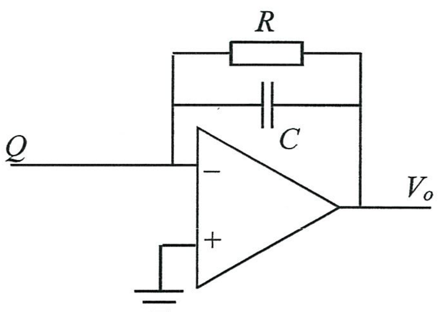
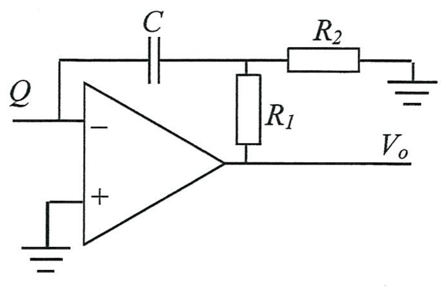
Equation of motion	Potential energy	Kinetic energy
$\rho A\dfrac{\partial^2 w}{\partial t^2} + EI\dfrac{\partial^4 w}{\partial x^4} = f(x,t)$	$V = \dfrac{1}{2}EI\int\left(\dfrac{\partial^2 w}{\partial x^2}\right)^2 dx$	$T = \dfrac{1}{2}\rho A\int\left(\dfrac{\partial w}{\partial t}\right)^2 dx$

Note that values of I can be found in the Mechanics Data Book.

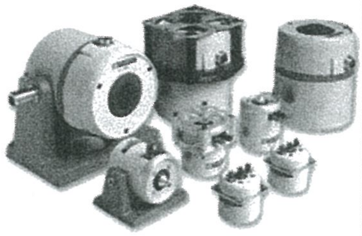
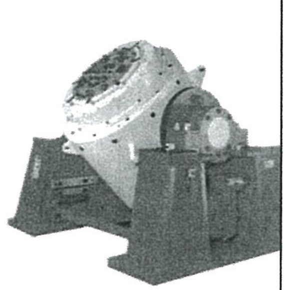
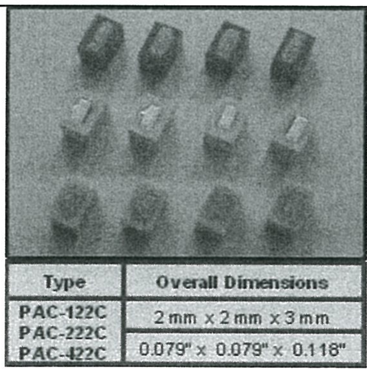
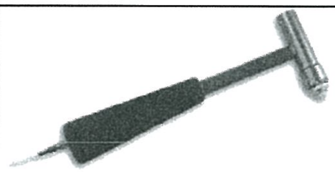
VIBRATION MEASUREMENT

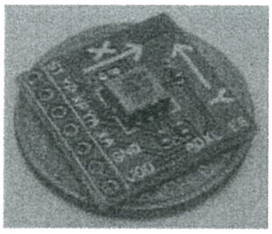
Some useful OpAmp circuits for instrumentation

(Note: j is used instead of i here for $\sqrt{-1}$ for compatibility with the Electrical Data Book.)

	Inverting voltage amplifier $V_o = -\frac{R_f}{R_i} V_i$
	Inverting voltage amplifier with low-pass and high-pass filter $V_o = -\frac{R_f}{R_i} \frac{V_i}{(1 + \frac{1}{j\omega R_i C_i})(1 + j\omega R_f C_f)}$
	Inverting charge amplifier $V_o = -\frac{Q}{C}$
	Inverting charge amplifier with high-pass filter $V_o = -\frac{Q}{C} \frac{1}{1 + \frac{1}{j\omega RC}}$
	Inverting charge amplifier with additional gain $V_o = -\frac{Q}{C} \frac{R_1 + R_2}{R_2}$

Some devices for vibration excitation and measurement

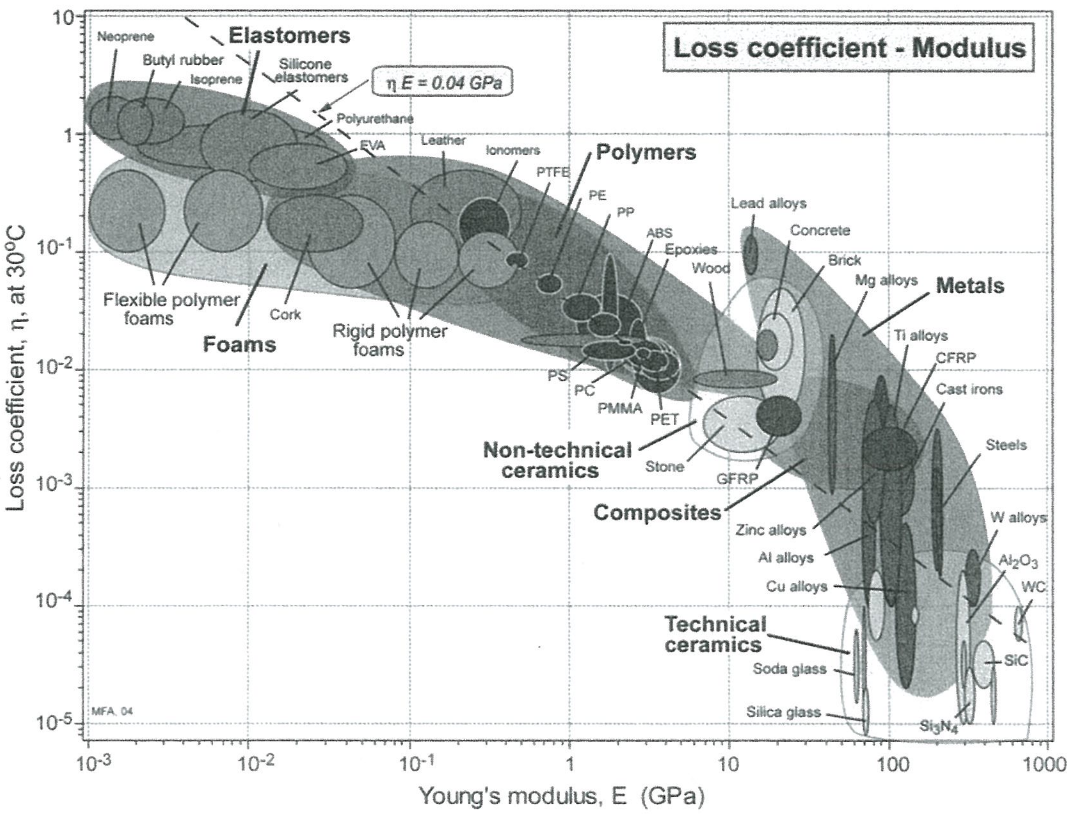
Moving coi electro-magnetic shaker	 LDS V101: Peak sine force 10N, internal armature resonance 12kHz Frequency range 5 – 12kHz, armature suspension stiffness 3.5N/mm, armature mass 6.5g, strol 2.5mm, shaker body mas 0.9kg	 LDS V650: Peak sine forc 1kN, internal armatur resonance 4kHz Frequency range 5 – 5kHz, armature suspension stiffness 16kN/m, armature mass 2.2kg, stroke 25mm, shaker body mass 200kg	 LDS V994: Peak sine force 300kN, internal armature resonance 1.4kHz Frequency range 5 – 1.7kHz, armature suspension stiffne 72kN/m, armatur mass 250kg, strol 50mm, shaker bo mass 13000kg
Piezo stack actuator		FACE PAC-122C Size 2×2×3mm Mass 0.1g Peak force 12N Stroke 1μm Unloaded resonance 400kH	
Impulse hammer		IH101 Head mass 0.1kg hammer tip stiffness 1500kN/m Force transducer sensitivit 4pC/N Internal resonance 50kHz	

Piezo accelero-meter		B&K4374 Mass 0.65g sensitivity 1.5pC/g, 1-26kHz, full-scale rang +/-5000g DJB A/23 Mass 5g, sensitivity 10pC/g, 1-20kHz, full-scale rang +/-2000g B&K4370 Mass 10g sensitivity 100pC/g, 1 4.8kHz, , full-scale range +/-2000g	
MEMS accelero-meter		ADKL202E 265mV/g Full scale range +/- 2g DC-6kHz	
Laser Doppler Vibrometer		Polytec PSV-400 Scannin Vibrometer Velocity ranges 2/10/50/100/1000 [mm/s/V]	

VIBRATION DAMPING

Correspondence principle

For linear viscoelastic materials, if an undamped problem can be solved then the corresponding solution to the damped problem is obtained by replacing the elastic moduli with complex values (which may depend on frequency): for example Young's modulus $E \rightarrow E(1+i\eta)$. Typical values of E and η for engineering materials are shown below:



Free and constrained layers

For a 2-layer beam: if layer j has Young's modulus E_j, second moment of area I_j and thickness h_j, the effective bending rigidity EI is given by:

$$EI = E_1 I_1 \left[1 + eh^3 + 3(1+h)^2 \frac{eh}{1+eh}\right]$$

where

$$e = \frac{E_2}{E_1}, \quad h = \frac{h_2}{h_1}.$$

For a 3-layer beam, using the same notation, the effective bending rigidity is

$$EI = E_1 \frac{h_1^3}{12} + E_2 \frac{h_2^3}{12} + E_3 \frac{h_3^3}{12} - E_2 \frac{h_2^2}{12}\left[\frac{h_{31} - d}{1+g}\right] + E_1 h_1 d^2 + E_2 h_2 (h_{21} - d)^2$$

$$+ E_3 h_3 (h_{31} - d)^2 - \left[\frac{E_2 h_2}{2}(h_{21} - d) + E_3 h_3 (h_{31} - d)\right]\left[\frac{h_{31} - d}{1+g}\right]$$

where $d = \dfrac{E_2 h_2 (h_{21} - h_{31}/2) + g(E_2 h_2 h_{21} + E_3 h_3 h_{31})}{E_1 h_1 + E_2 h_2 / 2 + g(E_1 h_1 + E_2 h_2 + E_3 h_3)}$,

$$h_{21} = \frac{h_1 + h_2}{2}, \quad h_{31} = \frac{h_1 + h_3}{2} + h_2, \quad g = \frac{G_2}{E_3 h_3 h_2 p^2},$$

G_2 is the shear modulus of the middle layer, and $p = 2\pi / (\text{wavelength})$, i.e. "wavenumber".

Viscous damping, the dissipation function and the first-order method

For a discrete system with viscous damping, then Rayleigh's dissipation function $F = \frac{1}{2}\underline{\dot{y}}^t C \underline{\dot{y}}$ is equal to half the rate of energy dissipation, where $\underline{\dot{y}}$ is the vector of generalised velocities (as on p.1), and C is the (symmetric) dissipation matrix.

If the system has mass matrix M and stiffness matrix K, free motion is governed by

$$M \underline{\ddot{y}} + C \underline{\dot{y}} + K \underline{y} = \underline{0}.$$

Modal solutions can be found by introducing the vector $\underline{z} = \begin{bmatrix} \underline{y} \\ \underline{\dot{y}} \end{bmatrix}$. If $\underline{z} = \underline{u} e^{\lambda t}$ then $\underline{u}$, λ are the eigenvectors and eigenvalues of the matrix

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}C \end{bmatrix}$$

where 0 is the zero matrix and I is the unit matrix.

THE HELMHOLTZ RESONATOR

A Helmholtz resonator of volume V with a neck of effective length L and cross-sectional area S has a resonant frequency

$$\omega = c\sqrt{\frac{S}{VL}}$$

where c is the speed of sound in air.

The end correction for an unflanged circular neck of radius a is $0.6a$.

The end correction for a flanged circular neck of radius a is $0.8a$.

VIBRATION OF A MEMBRANE

If a uniform plane membrane with tension T and mass per unit area m undergoes small transverse free vibration with displacement w, the motion is governed by the differential equation

$$T\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of Cartesian coordinates x, y or

$$T\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2}\right) = m\frac{\partial^2 w}{\partial t^2}$$

in terms of plane polar coordinates r,θ.

For a circular membrane of radius a the mode shapes are given by

$$\left.\begin{matrix}\sin\\ \cos\end{matrix}\right\} n\theta\, J_n(kr), \qquad n = 0,1,2,3\cdots$$

where J_n is the Bessel function of order n and k is determined by the condition that $J_n(ka) = 0$. The first few zeros of J_n's are as follows:

	$n=0$	$n=1$	$n=2$	$n=3$
$ka =$	2.404	3.832	5.135	6.379
$ka =$	5.520	7.016	8.417	9.760
$ka =$	8.654	10.173		

For a given k the corresponding natural frequency ω satisfies

$$k = \omega\sqrt{m/T}.$$