ENGINEERING TRIPOS PART IIB

Friday 13 May 2011 9.00 to 10.30

Module 4C9

CONTINUUM MECHANICS

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment:
4C9 datasheet (6 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

vsd03

1 The Kronecker delta and the permutation tensor are denoted by the symbols δ_{ij} and e_{ijk}, respectively.

(a) Determine the component f_2 for the following vector expressions

(i) $f_i = C_{ij} t_j$, [10%]

(ii) $f_i = e_{ijk} T_{jk}$. [10%]

(b) Using the $e - \delta$ identity, write $e_{pqs} e_{sqr}$ in terms of the Kronecker delta. [15%]

(c) Given the coordinate transformation law $A_{ij} = a_{pi} a_{qj} A'_{pq}$, where a_{mn} denotes the cosine of the angle between the m^{th} primed and the n^{th} unprimed coordinate axes, show that

(i) A_{ii},

(ii) $A_{ij} A_{ij}$ and

(iii) $e_{ijk} e_{kjp} A_{ip}$

are invariant under coordinate transformations. [45%]

(d) If B_{ij} is skew-symmetric (i.e. $B_{ij} = -B_{ji}$) and A_{ij} is symmetric (i.e. $A_{ij} = A_{ji}$) show that $A_{ij} B_{ij} = 0$. [20%]

2 (a) For a function $\phi(x,y)$ in Cartesian co-ordinates show that

$$\frac{\partial \phi}{\partial y} = \frac{\partial \phi}{\partial r}\sin\theta + \frac{\partial \phi}{\partial \theta}\frac{\cos\theta}{r}$$

where the polar coordinates (r,θ) are related to the Cartesian coordinates (x,y) via the usual relations $x = r\cos\theta$ and $y = r\sin\theta$. [20%]

(b) The Airy stress function

$$\phi = A\left(\theta + \frac{\sin 2\theta}{2}\right),$$

where A is a constant, has been proposed to determine the stress field in an elastic half-space subjected to a surface couple M as shown in Fig. 1(a). Show that the stresses σ_{rr}, $\sigma_{\theta\theta}$ and $\sigma_{r\theta}$ calculated from ϕ satisfy the boundary conditions and hence determine A in terms of M. [30%]

(c) We wish to determine the stress field in the case of two equal and opposite couples of magnitude M acting a small distance a apart as shown in Fig. 1(b). Using superposition, or otherwise, determine an appropriate Airy stress function in the limit of a vanishingly small a. (Hint: Use the co-ordinate transformation relation derived in part (a)). [50%]

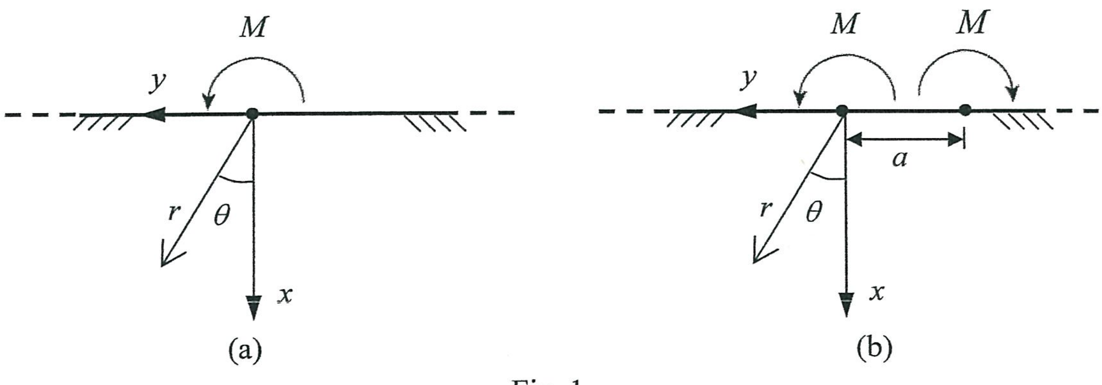

Fig. 1

(TURN OVER

3 (a) Explain the physical basis for the plastic deformation of metals. Why is volume conserved during plastic deformation? [15%]

(b) Volumetric plastic straining can occur in a porous metallic solid. Explain this observation. [15%]

(c) A thin-walled cylindrical tube of length l, diameter d and wall thickness t is made from an isotropic, elastic-plastic solid. The solid obeys the von Mises yield criterion and has an initial yield strength σ_Y. Strain hardening of this solid is linear such that in post-yield uniaxial tension, the tensile stress σ is related to the axial plastic strain ε^P by a constant plastic modulus $h = d\sigma / d\varepsilon^P$. Elastic straining is characterised by a Young's modulus E and Poisson ratio ν. The tube is subjected to an end tensile load P and torque Q which increase with time in fixed proportion $Q = \alpha P d$, where α is a dimensionless constant.

(i) Determine the stress state in the tube wall in terms of Q and P. [10%]

(ii) Obtain the yield load P_Y for $\alpha = 0$ and $\alpha = 1$. [20%]

(iii) For $\alpha = 1$, obtain an expression for the hardening rate dP / du at $P \geq P_Y$, where u is the axial extension. You may neglect the possibility of necking. [40%]

END OF PAPER

ENGINEERING TRIPOS Part IIB

Module 4C9 Data Sheet

SUBSCRIPT NOTATION

Repeated suffix implies summation

$\underset{\sim}{a} = a_1 \underset{\sim}{e}_1 + a_2 \underset{\sim}{e}_2 + a_3 \underset{\sim}{e}_3$	$a_i e_i$
$\underset{\sim}{a} \bullet \underset{\sim}{b}$	$a_i b_i \equiv a_i b_j \delta_{ij}$
$\underset{\sim}{c} = \underset{\sim}{a} \times \underset{\sim}{b}$	$c_i = e_{ijk} a_j b_k$
$\underset{\sim}{d} = \underset{\sim}{a} \times (\underset{\sim}{b} \times \underset{\sim}{c})$	$d_k = -e_{ijk} e_{irs} a_j b_r c_s = a_j b_k c_j - a_i b_i c_k$
Kronecker delta δ_{ij}	$\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$
e_{ijk}	$e_{ijk} = 1$ when indices cyclic; $= -1$ when indices anticyclic and $= 0$ when any indices repeat
$e - \delta$ identity	$e_{ijk} e_{ilm} \equiv \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$
trace$a$	$\mathrm{tr}\, a = a_{ii} = a_{11} + a_{22} + a_{33}$
$\dfrac{\partial \sigma_{ij}}{\partial x_i} = \dfrac{\partial \sigma_{1j}}{\partial x_1} + \dfrac{\partial \sigma_{2j}}{\partial x_2} + \dfrac{\partial \sigma_{3j}}{\partial x_3}$	$\sigma_{ij,i}$
gradϕ $= \nabla \phi$	$\dfrac{\partial \phi}{\partial x_i} = \phi_{,i}$
div$\underset{\sim}{V}$	$V_{i,i}$
curl$\underset{\sim}{V} \equiv \underset{\sim}{\nabla} \times \underset{\sim}{V}$	$e_{ijk} V_{k,j}$

Rotation of Orthogonal Axes

If $01'2'3'$ is related to 0123 by rotation matrix a_{ij}

vector v_i becomes	$v'_\alpha = a_{\alpha i} v_i$
tensor σ_{ij} becomes	$\sigma'_{\alpha\beta} = a_{\alpha i} a_{\beta j} \sigma_{ij}$

Evaluation of principal stresses

deviatoric stress $s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}$

$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = 0$

$$I_1 = \sigma_{ii} = \mathrm{tr}\sigma$$

$$I_2 = \frac{1}{2}\left(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ij}\right)$$

$$I_3 = \frac{1}{6}\left(e_{ijk}e_{pqr}\sigma_{ip}\sigma_{jq}\sigma_{kr}\right)$$

$s^3 - J_2 s - J_3 = 0$

$$J_1 = s_{ii} = \mathrm{tr}s \; ; \; J_2 = \frac{1}{2}s_{ij}s_{ij} \; ; \; J_3 = \frac{1}{3}s_{ij}s_{jk}s_{ki}$$

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equilibrium $\qquad \sigma_{ij,i} + b_j = 0$

small strains
$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \equiv \frac{1}{2}\left(u_{i,j} + u_{j,i}\right)$$

compatibility
$$\varepsilon_{ij,kl} + \varepsilon_{kl,ij} - \varepsilon_{lj,ki} - \varepsilon_{ki,lj} e_{pik}e_{qjl}\varepsilon_{ij,kl} = 0$$

equivalent to $\quad e_{pik}e_{qjl}\varepsilon_{ij,kl} \equiv e_{pik}e_{qjl}\dfrac{\partial^2 \varepsilon_{ij}}{\partial x_k \partial x_l} = 0$

Linear elasticity $\qquad \sigma_{ij} = C_{ijkl}\varepsilon_{kl}$

Hooke's law $\qquad E\varepsilon_{ij} = (1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}$

Lamé's equations $\qquad \sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}$

von Mises equivalent stress $\qquad \sigma_e \equiv \bar{\sigma} = \sqrt{\frac{3}{2}s_{ij}s_{ij}} = \sqrt{3J_2}$

equivalent strain increment $\qquad \mathrm{d}\bar{\varepsilon} = \sqrt{\frac{2}{3}\mathrm{d}\varepsilon_{ij}\mathrm{d}\varepsilon_{ij}}$

Elastic torsion of prismatic bars

Warping function $\Psi(x_1, x_2)$ satisfies $\nabla^2\Psi = \Psi_{,ii} = 0$

If Prandtl stress function $\phi(x_1, x_2)$ satisfies $\nabla^2\phi = \phi_{,ii} = -2G\alpha$ where α is the twist per unit length then

$$\sigma_{31} = \phi_{,2} = \frac{\partial\phi}{\partial x_2} \; , \; \sigma_{32} = -\phi_{,1} = -\frac{\partial\phi}{\partial x_1} \quad \text{and} \quad T = 2\iint_A \phi(x_1, x_2)\mathrm{d}x_1\mathrm{d}x_2$$

Equivalence of elastic constants

	E	ν	$G=\mu$	λ
E, ν	–	–	$\frac{E}{2(1+\nu)}$	$\frac{\nu E}{(1+\nu)(1-2\nu)}$
E, G	–	$\frac{E-2G}{2G}$	–	$\frac{(2G-E)G}{E-3G}$
E, λ	–	$\frac{E-\lambda+R}{4\lambda}$	$\frac{E-3\lambda+R}{4}$	–
ν, G	$2G(1+\nu)$	–	–	$\frac{2G\nu}{1-2\nu}$
ν, λ	$\frac{\lambda(1+\nu)(1-2\nu)}{\nu}$	–	$\frac{\lambda(1-2\nu)}{2\nu}$	–
G, λ	$\frac{G(3\lambda+2G)}{\lambda+G}$	$\frac{\lambda}{2(\lambda+G)}$	–	–

$$R=\sqrt{E^2+2E\lambda+9\lambda^2}$$

Two-dimensional Airy Stress function

Biharmonic equation $\nabla^4\phi \equiv \phi_{,\alpha\alpha\beta\beta}=0$

Stresses $\sigma_{\alpha\beta}=e_{\alpha\gamma}\, e_{\beta\delta}\, \phi_{,\gamma\delta}$

$$\text{where } e_{\alpha\beta} \equiv e_{3\alpha\beta} = \begin{cases} 1 \text{ if } \alpha=1, \beta=2 \\ 0 \text{ if } \alpha=\beta \\ -1 \text{ if } \alpha=2, \beta=1 \end{cases}$$

Plane stress and plane strain

$$G\varepsilon_{11}=\frac{1}{8}\{\sigma_{11}(1+\kappa)+\sigma_{22}(\kappa-3)\}$$

$$G\varepsilon_{22}=\frac{1}{8}\{\sigma_{22}(1+\kappa)+\sigma_{11}(\kappa-3)\}$$

$$G\varepsilon_{12}=\frac{\sigma_{12}}{2}$$

$$\text{where } \begin{cases} \kappa=(3-\nu)/(1+\nu) & \text{in plane stress and} \\ \kappa=3-4\nu & \text{in plane strain} \end{cases}$$

Plasticity

von Mises yield criterion $f = \sigma_e - Y = 0$

J_2 flow rule $\dot{\varepsilon}_{ij}^{PL} = \frac{3}{2}\frac{s_{ij}}{\sigma_e}\frac{\dot{\sigma}_e}{h}$

Slip Line Fields

Henky equations

$p + 2k\phi =$ constant along α line

$p - 2k\phi =$ constant along β line

Geiringer equations

$\frac{dv_\alpha}{ds} = v_\beta \frac{d\phi}{ds}$ along α line

$\frac{dv_\beta}{ds} = -v_\alpha \frac{d\phi}{ds}$ along β line

Table I – The Michell solutions — stress components

$\phi(r,\theta)$	σ_{rr}	$\sigma_{\theta\theta}$	$\sigma_{r\theta}$
r^2	2	2	0
$r^2 \ln r$	$2\ln r + 1$	$2\ln r + 3$	0
$\ln r$	$1/r^2$	$-1/r^2$	0
θ	0	0	$1/r^2$
$r^3 \cos\theta$	$2r\cos\theta$	$6r\cos\theta$	$2r\sin\theta$
$r\theta\sin\theta$	$2\cos\theta/r$	0	0
$r\ln r\cos\theta$	$\cos\theta/r$	$\cos\theta/r$	$\sin\theta/r$
$\cos\theta/r$	$-2\cos\theta/r^3$	$2\cos\theta/r^3$	$-2\sin\theta/r^3$
$r^3 \sin\theta$	$2r\sin\theta$	$6r\sin\theta$	$-2r\cos\theta$
$r\theta\cos\theta$	$-2\sin\theta/r$	0	0
$r\ln r\sin\theta$	$\sin\theta/r$	$\sin\theta/r$	$-\cos\theta/r$
$\sin\theta/r$	$-2\sin\theta/r^3$	$2\sin\theta/r^3$	$2\cos\theta/r^3$
$r^{n+2}\cos n\theta$	$-(n+1)(n-2)r^n \cos n\theta$	$(n+1)(n+2)r^n \cos n\theta$	$n(n+1)r^n \sin n\theta$
$r^{-n+2}\cos n\theta$	$-(n+2)(n-1)r^{-n} \cos n\theta$	$(n-1)(n-2)r^{-n} \cos n\theta$	$-n(n-1)r^{-n} \sin n\theta$
$r^n \cos n\theta$	$-n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \cos n\theta$	$n(n-1)r^{n-2} \sin n\theta$
$r^{-n}\cos n\theta$	$-n(n+1)r^{-n-2} \cos n\theta$	$n(n+1)r^{-n-2} \cos n\theta$	$-n(n+1)r^{-n-2} \sin n\theta$
$r^{n+2}\sin n\theta$	$-(n+1)(n-2)r^n \sin n\theta$	$(n+1)(n+2)r^n \sin n\theta$	$-n(n+1)r^n \cos n\theta$
$r^{-n+2}\sin n\theta$	$-(n+2)(n-1)r^{-n} \sin n\theta$	$(n-1)(n-2)r^{-n} \sin n\theta$	$n(n-1)r^{-n} \cos n\theta$
$r^n \sin n\theta$	$-n(n-1)r^{n-2} \sin n\theta$	$n(n-1)r^{n-2} \sin n\theta$	$-n(n-1)r^{n-2} \cos n\theta$
$r^{-n}\sin n\theta$	$-n(n+1)r^{-n-2} \sin n\theta$	$n(n+1)r^{-n-2} \sin n\theta$	$n(n+1)r^{-n-2} \cos n\theta$

Table II – The Michell solutions — displacement components

For plane strain $\kappa = 3-4\nu$; for plane stress $\kappa = (3-\nu)/(1+\nu)$

$\phi(r,\theta)$	$2Gu_r$	$2Gu_\theta$
r^2	$(\kappa-1)r$	0
$r^2\ln r$	$(\kappa-1)r\ln r - r$	$(\kappa+1)r\theta$
$\ln r$	$-1/r$	0
θ	0	$-1/r$
$r^3\cos\theta$	$(\kappa-2)r^2\cos\theta$	$(\kappa+2)r^2\sin\theta$
$r\theta\sin\theta$	$0.5[(\kappa-1)\theta\sin\theta-\cos\theta + (\kappa+1)\ln r\cos\theta]$	$0.5[(\kappa-1)\theta\cos\theta-\sin\theta - (\kappa+1)\ln r\sin\theta]$
$r\ln r\cos\theta$	$0.5[(\kappa+1)\theta\sin\theta-\cos\theta + (\kappa-1)\ln r\cos\theta]$	$0.5[(\kappa+1)\theta\cos\theta-\sin\theta - (\kappa-1)\ln r\sin\theta]$
$\cos\theta/r$	$\cos\theta/r^2$	$\sin\theta/r^2$
$r^3\sin\theta$	$(\kappa-2)r^2\sin\theta$	$-(\kappa-2)r^2\cos\theta$
$r\theta\cos\theta$	$0.5[(\kappa-1)\theta\cos\theta+\sin\theta - (\kappa+1)\ln r\sin\theta]$	$0.5[-(\kappa-1)\theta\sin\theta-\cos\theta - (\kappa+1)\ln r\cos\theta]$
$r\ln r\sin\theta$	$0.5[-(\kappa+1)\theta\cos\theta-\sin\theta + (\kappa-1)\ln r\sin\theta]$	$0.5[(\kappa+1)\theta\sin\theta+\cos\theta + (\kappa-1)\ln r\cos\theta]$
$\sin\theta/r$	$\sin\theta/r^2$	$-\cos\theta/r^2$
$r^{n+2}\cos n\theta$	$(\kappa-n-1)r^{n+1}\cos n\theta$	$(\kappa+n+1)r^{n+1}\sin n\theta$
$r^{-n+2}\cos n\theta$	$(\kappa+n-1)r^{-n+1}\cos n\theta$	$-(\kappa-n+1)r^{-n+1}\sin n\theta$
$r^n\cos n\theta$	$-nr^{n-1}\cos n\theta$	$nr^{n-1}\sin n\theta$
$r^{-n}\cos n\theta$	$nr^{-n-1}\cos n\theta$	$nr^{-n-1}\sin n\theta$
$r^{n+2}\sin n\theta$	$(\kappa-n-1)r^{n+1}\sin n\theta$	$-(\kappa+n+1)r^{n+1}\cos n\theta$
$r^{-n+2}\sin n\theta$	$(\kappa+n-1)r^{-n+1}\sin n\theta$	$(\kappa-n+1)r^{-n+1}\cos n\theta$
$r^n\sin n\theta$	$-nr^{n-1}\sin n\theta$	$-nr^{n-1}\cos n\theta$
$r^{-n}\sin n\theta$	$nr^{-n-1}\sin n\theta$	$-nr^{-n-1}\cos n\theta$

JAW//NAF

Numerical answers to 4C[illegible](2011)

1. (a)

(i) $f_2 = C_{21}t_1 + C_{22}t_2 + C_{23}t_3$

(ii) $T_{31} - T_{13}$

(b) $-2\delta_{pr}$

2. (b) $A = \dfrac{M}{\pi}$

(c) $\phi_1 = -\dfrac{2Ma}{\pi r}\cos^3\theta$

3.

(c)

(i)

$$\sigma_a = \frac{P}{\pi dt}$$

$$\tau = \frac{2Q}{\pi d^2 t}$$

(ii)

$$P_Y = \sigma_Y \pi dt$$

$$P_Y = \frac{\sigma_Y \pi dt}{\sqrt{13}}$$

(iii) $\dfrac{dP}{du} = \dfrac{\pi dth}{l}$