ENGINEERING TRIPOS PART IIB

Friday 13 May 2011 9 to 10.30

Module 4F1

CONTROL SYSTEM DESIGN

*Answer not more than **two** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Formulae sheet (3 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed
Supplementary pages: Two extra copies of Fig. 2 (Question 2).

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

1 (a) Let $G(s)$ be a rational transfer-function.

(i) What is the definition of the root-locus diagram of $G(s)$? [10%]

(ii) State the meaning of breakaway points of the root-locus. [7%]

(iii) Suppose it is known that the root-locus passes through a point s_0 for some gain k_0. Explain how k_0 can be found. [8%]

(b) The equations of motion for the quarter-car vehicle suspension model of Fig. 1 are given by:

$$\begin{aligned} m_s\ddot{x} &= c(\dot{y}-\dot{x})+k(y-x), \\ m_u\ddot{y} &= c(\dot{x}-\dot{y})+k(x-y)-k_t y, \end{aligned}$$

where m_s, m_u are the sprung and unsprung masses, k_t, k are the tyre and suspension spring constants and c is the damper constant.

(i) Show that the characteristic equation of the suspension system can be written in the form [15%]

$$\left(m_s m_u s^4+(m_s k+m_s k_t+m_u k)s^2+kk_t\right)+cs\left((m_s+m_u)s^2+k_t\right)=0.$$

(ii) Suppose the mass and spring constants are kept fixed and c is varied from 0 to ∞. Explain how the locus of characteristic roots can be interpreted as a root-locus diagram and find the relevant transfer function $G(s)$. [10%]

(iii) Suppose $m_u = 1$, $m_s = 8$, $k = 8$ and $k_t = 72$ in some system of units. Show that $G(s)$ has its poles and zeros in alternating locations on the imaginary axis. [10%]

(iv) The breakaway points for this $G(s)$ are located at $s = \pm 1.31$, ± 2.64, ± 6.95. Sketch the root-locus diagram for c varying between 0 and ∞. [20%]

(v) Find the range of values of $c > 0$ for which all the roots are on the real axis. [20%]

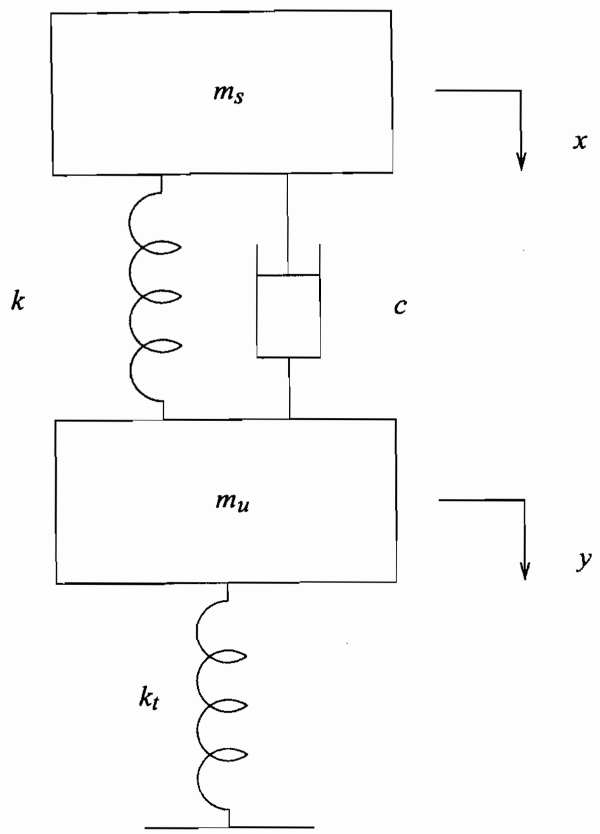


Fig. 1

(TURN OVER

2 (a) Briefly describe how the decomposition of a time function $q(t)$ into even and odd functions can lead to a derivation of the Bode gain-phase relationships. (Mathematical derivation not required.) [15%]

(b) Fig. 2 is the Bode diagram of a system $G(s)$ which has no poles satisfying $\mathrm{Re}(s) > 0$.

(i) Sketch on a copy of Fig. 2 the expected phase of $G(j\omega)$ if $G(s)$ were stable and minimum phase. [10%]

(ii) What does this plot suggest about possible zeros of $G(s)$ satisfying $\mathrm{Re}(s) \geq 0$? [15%]

(c) A feedback compensator $K(s)$ in the standard negative feedback configuration is to be designed for the system $G(s)$ whose Bode diagram is shown in Fig. 2. The following specifications are sought for the return ratio $L(s) = G(s)K(s)$:

A: Phase margin of at least 45°;

B: $|L(j\omega)| \geq 10$ for $\omega \leq \omega_1$;

C: $|L(j\omega)| \leq 0.1$ for $\omega \geq \omega_2$

for some frequencies ω_1 and ω_2.

(i) Design a compensator $K(s)$ to achieve the above specifications for $\omega_1 = 0.1$ rad s^{-1} and $\omega_2 = 10$ rad s^{-1}. Show the Bode diagram of your compensator and the resulting return ratio on a copy of Fig. 2. [35%]

(ii) Explain why it would be difficult to design a compensator $K(s)$ to achieve the above specifications for $\omega_1 = 1$ rad s^{-1} and $\omega_2 = 10$ rad s^{-1}. [25%]

Two copies of Fig. 2 are provided on separate sheets. These should be handed in with your answers.

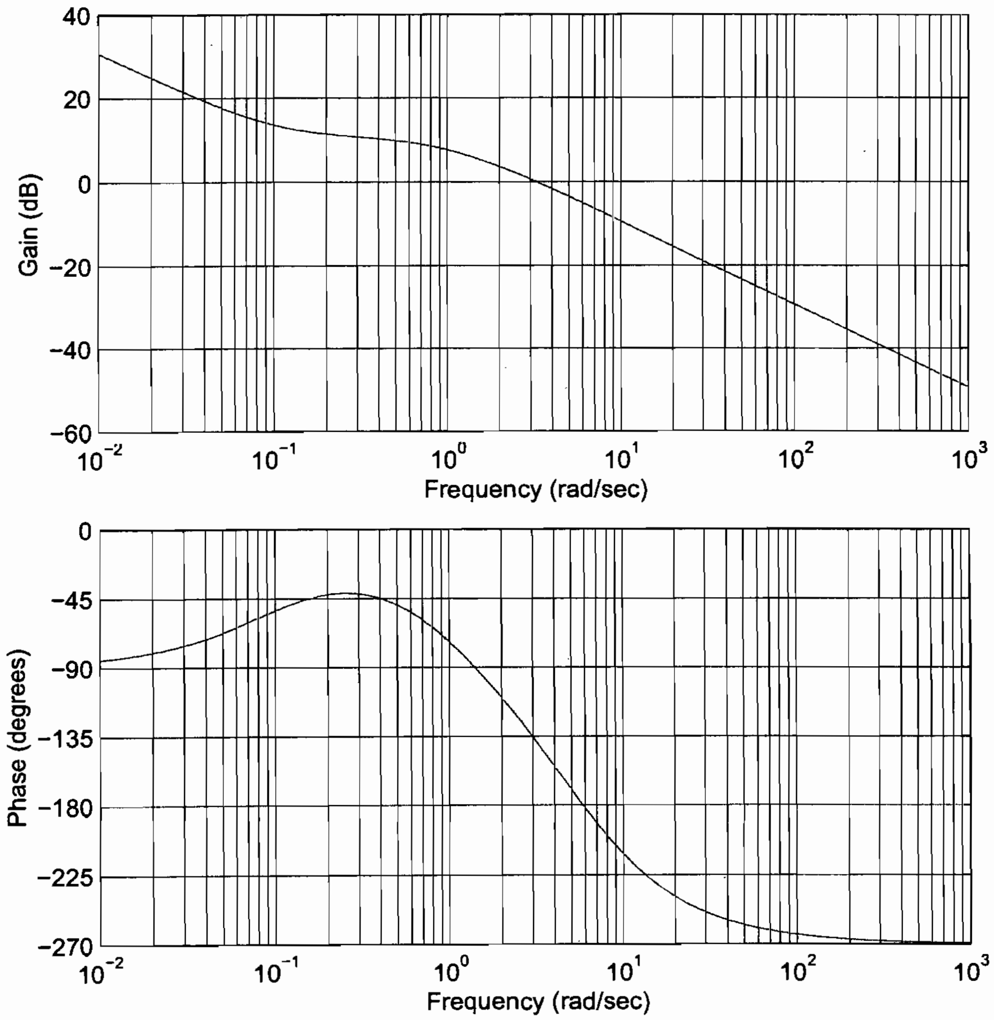

40
20
0
-20
-40
-60
Gain (dB)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)
0
-45
-90
-135
-180
-225
-270
Phase (degrees)
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}
10^{3}
Frequency (rad/sec)

Fig. 2

3 A control system is to be designed for a plant whose transfer-function model is:

$$G(s) = \frac{s-1}{(s-3)(s+2)}.$$

(a) Explain briefly the difficulties which would be expected in controlling this plant. [15%]

(b) A controller $K(s) = K_1(s)$ is proposed where

$$K_1(s) = \frac{8(s+2)}{(s-3)}.$$

Verify that this controller is stabilising when connected in the standard unity gain negative feedback configuration. [15%]

(c) It is desired to asymptotically reject sensor noise at the frequency 20 rad s^{-1}. Accordingly the controller $K(s) = K_1(s)K_2(s)$ is proposed with

$$K_2(s) = \frac{s^2+400}{s^2+cs+400}$$

for some $c > 0$. By writing $G(s)K(s) = G(s)K_1(s)(1+\Delta(s))$, or otherwise, show that the feedback system will remain stable providing

$$\left|\frac{8c\omega}{-\omega^2+cj\omega+400}\right| < |1+j\omega| \qquad (1)$$

for all ω. State clearly any results you use. [25%]

(d) Using Bode diagrams, or otherwise, verify that (1) is satisfied when $c = 40$. [20%]

(e) Design a two-degree-of-freedom control system to achieve the specification of Part (c) and a transfer function relating plant output to reference input equal to

$$\frac{1-s}{(5s+1)^2}.$$

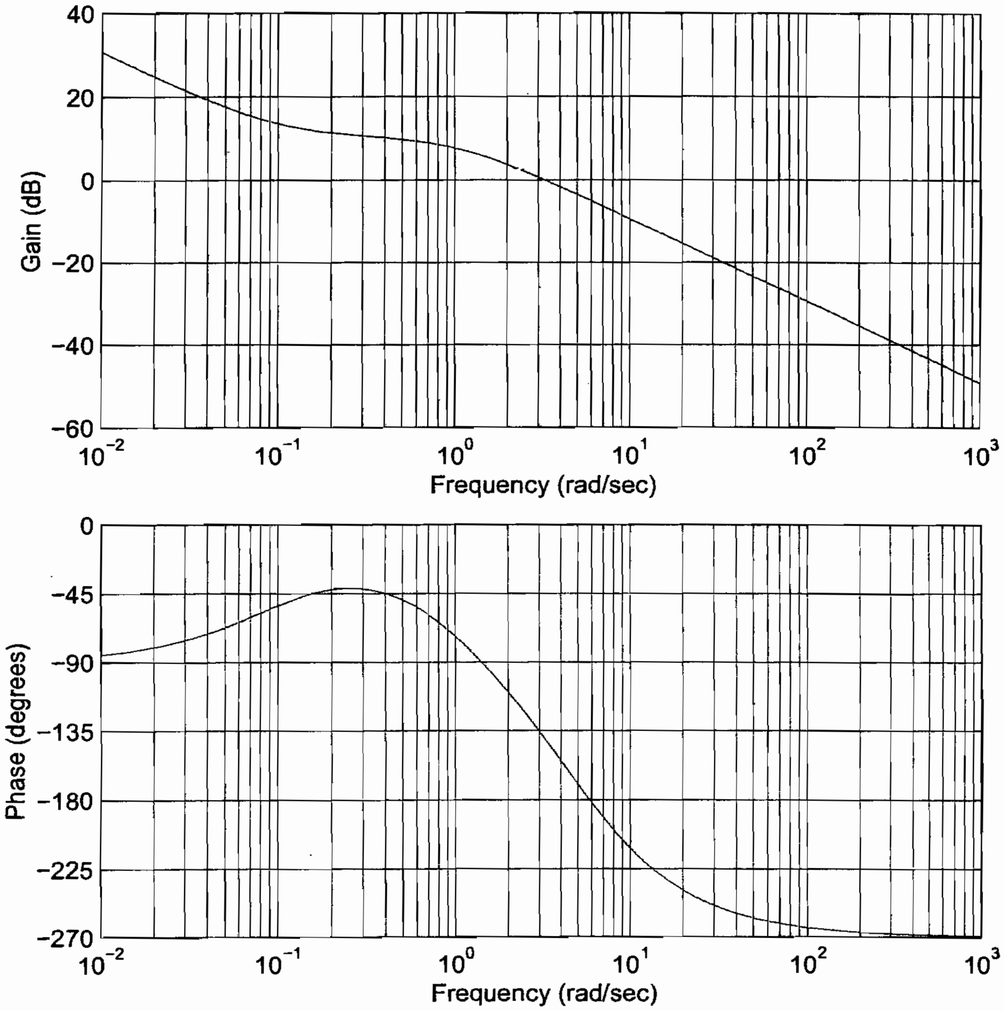
[25%]

END OF PAPER

Candidate Number:

ENGINEERING TRIPOS PART IIB

Friday 13 May 2011, Module 4F1, Question 2.



Extra copy of Fig. 2: Bode diagram of $G(s)$ for Question 2.

Formulae sheet for Module 4F1: Control System Design

To be available during the examination.

1 Terms

For the standard feedback system shown below, the **Return-Ratio Transfer Function** $L(s)$ is given by

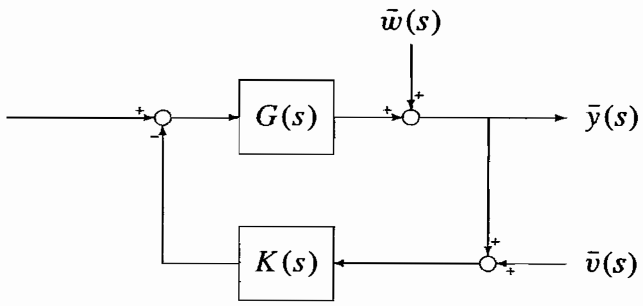
$$L(s) = G(s)K(s),$$

the **Sensitivity Function** $S(s)$ is given by

$$S(s) = \frac{1}{1 + G(s)K(s)}$$

and the **Complementary Sensitivity Function** $T(s)$ is given by

$$T(s) = \frac{G(s)K(s)}{1 + G(s)K(s)}$$



The closed-loop system is called **Internally Stable** if each of the *four* closed-loop transfer functions

$$\frac{1}{1 + G(s)K(s)}, \quad \frac{G(s)K(s)}{1 + G(s)K(s)}, \quad \frac{K(s)}{1 + G(s)K(s)}, \quad \frac{G(s)}{1 + G(s)K(s)}$$

are stable (which is equivalent to $S(s)$ being stable and there being no right half plane pole/zero cancellations between $G(s)$ and $K(s)$).
A transfer function is called **real-rational** if it can be written as the ratio of two polynomials in s, the coefficients of each of which are purely real.

2 Phase-lead compensators

The phase-lead compensator

$$K(s) = \alpha \frac{s + \omega_c/\alpha}{s + \omega_c \alpha}, \qquad \alpha > 1$$

achieves its maximum phase advance at $\omega = \omega_c$, and satisfies:

$$|K(j\omega_c)| = 1, \quad \text{and} \quad \angle K(j\omega_c) = 2 \arctan \alpha - 90^\circ.$$

3 The Bode Gain/Phase Relationship

If

1. $L(s)$ is a real-rational function of s,
2. $L(s)$ has no poles or zeros in the *open* RHP ($\text{Re}(s) > 0$) and
3. satisfies the normalization condition $L(0) > 0$.

then

$$\boxed{\angle L(j\omega_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{d}{dv}\log|L(j\omega_0 e^v)| \log\coth\frac{|v|}{2}\,dv}$$

Note that

$$\log\coth\frac{|v|}{2} = \log\left|\frac{\omega+\omega_0}{\omega-\omega_0}\right|, \text{ where } \omega = \omega_0 e^v.$$

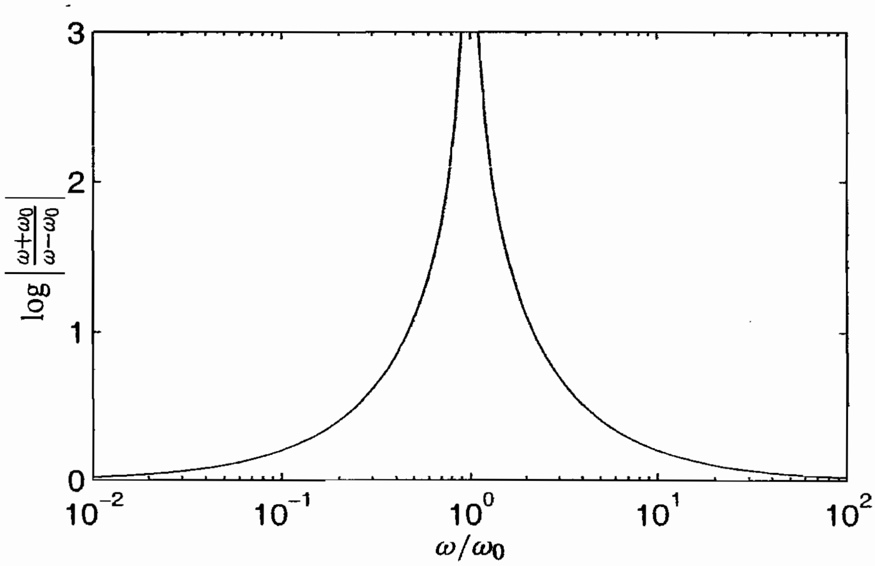


Figure 1:

If the slope of $L(j\omega)$ is approximately constant for a sufficiently wide range of frequencies around $\omega = \omega_0$ we get the *approximate form of the Bode Gain/Phase Relationship*

$$\angle L(j\omega_0) \approx \frac{\pi}{2}\left.\frac{d\log|L(j\omega_0 e^v|)}{dv}\right|_{\omega=\omega_0}.$$

4 The Poisson Integral

If $H(s)$ is a real-rational function of s which has no poles or zeros in $\mathrm{Re}(s) > 0$, then if $s_0 = \sigma_0 + j\omega_0$ with $\sigma_0 > 0$

$$\boxed{\log H(s_0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sigma_0}{\sigma_0^2 + (\omega - \omega_0)^2} \log H(j\omega)\, d\omega}$$

and

$$\boxed{\log |H(s_0)| = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} \log \left|H\left(j|s_0|e^{v}\right)\right| dv}$$

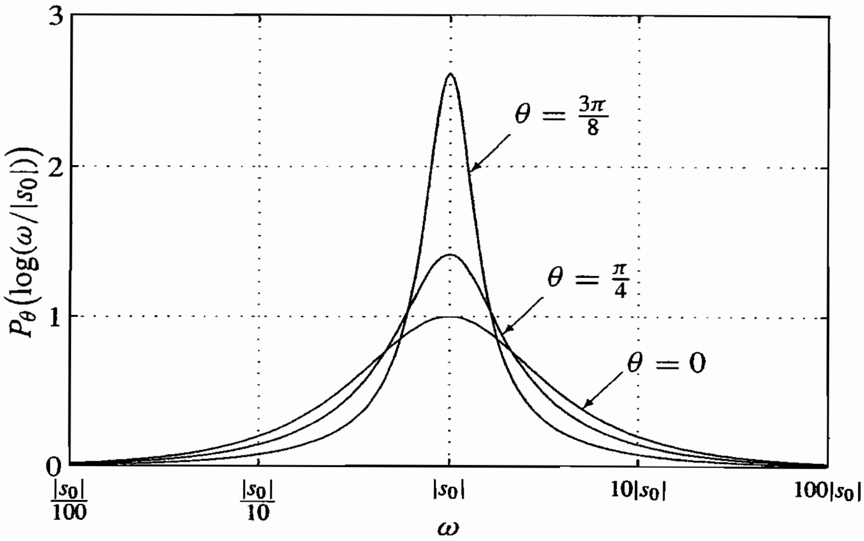
where $v = \log\left(\frac{\omega}{|s_0|}\right)$ and $\theta = \angle(s_0)$. Note that, if s_0 is real, so $\angle s_0 = 0$, then

$$\frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta} = \frac{1}{\cosh v}.$$

We define

$$P_\theta(v) = \frac{\cosh v \cos\theta}{\sinh^2 v + \cos^2\theta}$$

and give graphs of P_θ below.



The indefinite integral is given by

$$\int P_\theta(v)\, dv = \arctan\left(\frac{\sinh v}{\cos\theta}\right)$$

and

$$\frac{1}{\pi}\int_{-\infty}^{\infty} P_\theta(v)\, dv = 1 \quad \text{for all } \theta.$$

G. Vinnicombe
M.C. Smith
November 2002

4F1 2011 — Answers

1(e) $14.94 < c < 15.41$.

2(b)(ii) $G(s)$ has a RHP zero around $s = 5$.

M.C. Smith, 6 June 2011