ENGINEERING TRIPOS PART IIB

Friday 6 May 2011 2.30 to 4.00

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

There are no attachments.

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

sss01

1 (a) A plant H has input $\{v(n)\}_{n=0}^{\infty}$ and output $\{z(n)\}_{n=0}^{\infty}$. The aim is to design the best fitting inverse model to the unknown H. Describe how (with the aid of a diagram) an adaptive filter may be applied to this problem. [30%]

(b) Consider the following gradient descent algorithm

$$\mathbf{h}(n) = \mathbf{h}(n-1) + \mu \mathbf{R}^{-1}(\mathbf{p} - \mathbf{R}\mathbf{h}(n-1)), \quad n > 0, \qquad (1)$$

where $\mathbf{R}$ is a positive definite square matrix, $\mathbf{h}(0)$ and $\mathbf{p}$ are vectors, and μ is a scalar.

Assuming $\mathbf{h}(n)$ converges to a limit $\mathbf{h}^*$, derive the expression for $\mathbf{h}^*$. [15%]

(c) Deduce conditions on μ that would ensure $\mathbf{h}(n)$ converges regardless of the initial vector $\mathbf{h}(0)$ chosen. State and justify the optimal choice for μ. [25%]

(d) Let $\{u(n)\}_{n=0}^{\infty}$ be the input signal, $\{d(n)\}_{n=0}^{\infty}$ the reference signal, both stationary and ergodic, and

$$\mathbf{R} = E\{\mathbf{u}(n)\mathbf{u}(n)^{\mathrm{T}}\}, \qquad \mathbf{p} = E\{\mathbf{u}(n)d(n)\}.$$

In practice $\mathbf{R}$ and $\mathbf{p}$ are unknown and have to be estimated from the input and reference signal. Detail a modification of the Least Mean Square algorithm to implement equation (1). Contrast the behaviour of this algorithm with that of the Least Mean Square algorithm. [20%]

(e) Suggest a modification of the algorithm you proposed in part (d) if $E\{\mathbf{u}(n)\mathbf{u}(n)^{\mathrm{T}}\}$ and $E\{\mathbf{u}(n)d(n)\}$ are now slowly changing over time. [10%]

2 The signal $u(n)$ is generated as follows

$$u(n) = a\,u(n-1) + v(n) + b\,v(n-1)$$

where $v(n)$ are independent and identically distributed random variables with zero mean and variance σ_v^2. The aim is to design an m-th order predictor for $u(n)$

$$\widehat{u}(n) = h_0 u(n-1) + h_1 u(n-2) + \cdots + h_{m-1} u(n-m).$$

(a) Let $r_k = E\{u(n)u(n-k)\}$. Write down the simultaneous equations for r_0, r_1 and r_2 in terms of a, b and σ_v^2. [40%]

(b) Show that

$$J_{\min} = \min_{\mathbf{h}} E\left\{ \left(d(n) - \mathbf{h}^{\mathrm{T}}\mathbf{u}(n)\right)^2 \right\} = \sigma_d^2 - \mathbf{p}^{\mathrm{T}}\mathbf{R}^{-1}\mathbf{p}$$

where $\sigma_d^2 = E\{d(n)^2\}$, square matrix $\mathbf{R} = E\{\mathbf{u}(n)\mathbf{u}(n)^{\mathrm{T}}\}$ and vector $\mathbf{p} = E\{\mathbf{u}(n)d(n)\}$. Give the expression for the minimising $\mathbf{h}$. [30%]

(c) Given $a = 0.5$, $b = 1$ and $\sigma_v^2 = 1$, calculate $J_{\min}$ for a first and second order predictor. Discuss your calculated results. [30%]

3 (a) Discuss the periodogram method for power spectrum estimation. Your discussion should include issues concerning bias, variance, spectral resolution and improvements to the periodogram. [40%]

(b) A zero-mean random process has autocorrelation $R_{XX}[k] = \alpha^k$ for $|k| < 2$, and zero otherwise, where $|\alpha| < 1$. Compute the expected value of the periodogram estimate

$$\widehat{S}_X(e^{j\omega}) = \frac{1}{N}\left|\sum_{n=0}^{N-1} x_n e^{-j\omega n}\right|^2 .$$

Compare this expected value with the true power spectrum density. [40%]

(c) Repeat part (b) for the process

$$x_n = w_n + w_{n-1}$$

where w_n are independent random variables with zero mean and variance 0.5. [20%]

4 Let $x_n = ax_{n-1} + w_n$ where w_n are independent and identically distributed Gaussian random variables with mean zero and variance σ^2. Additionally, $|a| < 1$ and x_n is stationary.

(a) Write down the density $p(x_0)$ and the conditional density $p(x_1, \dots, x_n | x_0)$. [20%]

(b) Using $p(x_1, \dots, x_n | x_0)$, solve for the maximum likelihood estimates of a and σ^2. [40%]

(c) Relate the estimates obtained in (b) to those obtained by the Yule-Walker method. [15%]

(d) Describe a procedure to obtain the density of $(x_0, \dots, x_{P-1})$ where x_n is a stationary AR(P) process given by

$$x_n = \sum_{i=1}^{P} a_i x_{n-i} + w_n.$$

If the number of data points n is much larger than P, describe a simplified solution to the maximum likelihood problem. [25%]

END OF PAPER