ENGINEERING TRIPOS PART IIA
ENGINEERING TRIPOS PART IIB

Monday 2 May 2011 2.30 to 4

Module 4M12

PARTIAL DIFFERENTIAL EQUATIONS AND VARIATIONAL METHODS

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: 4M12 Data Sheet (3 sides).

STATIONERY REQUIREMENTS	SPECIAL REQUIREMENTS
Single-sided script paper	Engineering Data Book
	CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

Version: gnw05

1 For a body V with boundary ∂V and with outward unit normal vector $\boldsymbol{n}$ on ∂V:

(a) Use index notation to show that

$$\int_{\partial V} \boldsymbol{u} \cdot \boldsymbol{A}\boldsymbol{n} \, dS = \int_V \nabla \cdot (\boldsymbol{A}^T \boldsymbol{u}) \, dV$$

where $\boldsymbol{u}$ is a vector and $\boldsymbol{A}$ is a second-order tensor. [10%]

(b) If V is subject to tractions $\boldsymbol{t}$ on its boundary and a body force $\rho \boldsymbol{g}$, force equilibrium requires that

$$\nabla \cdot \boldsymbol{\sigma} + \rho \boldsymbol{g} = \boldsymbol{0}$$

where the stress $\boldsymbol{\sigma}$ is a second-order tensor and is defined by $\boldsymbol{\sigma n} = \boldsymbol{t}$ on ∂V. Write down an expression for moment equilibrium, and show that if force equilibrium is satisfied then moment equilibrium is satisfied if $\boldsymbol{\sigma} = \boldsymbol{\sigma}^T$. [50%]

(c) Determine the partial differential equation that is satisfied by w if w minimises the functional

$$J = \int_V \left(\nabla^2 w\right)^2 dV$$

Comment on the boundary conditions. [40%]

2 The steady-state heat equation with variable conductivity $\kappa = \kappa(\boldsymbol{x}) > 0$ and a prescribed source term f for a body V reads

$$-\nabla \cdot \kappa \nabla u = f$$

Boundary conditions $u = 0$ on ∂V are applied.

(a) Formulate a weak version of the steady-state heat equation. [15%]

(b) Derive a functional J whose minimum corresponds to a solution of the steady-state heat equation. Comment on the significance of the sign of the conductivity. [15%]

(c) Suppose that the heat equation can be controlled by varying κ such that $u \approx z$, where z is prescribed.

(i) Using the framework of optimal control, formulate a suitable functional whose stationary points correspond to the optimal control of this problem. [20%]

(ii) Find the system of partial differential equations that solve the optimal control problem. Comment on the type of equations that must be solved. [50%]

3 A second-order linear PDE is defined as

$$a\frac{\partial^2 u}{\partial x^2}+2b\frac{\partial^2 u}{\partial x\partial y}+c\frac{\partial^2 u}{\partial y^2}+d\left(\frac{\partial u}{\partial x},\frac{\partial u}{\partial y},u,x,y\right)=0$$

where the coefficients a, b and c depend on x and y.

(a) If a change of variables $\xi=\xi(x,y)$ and $\eta=\eta(x,y)$ is made, the PDE reads

$$A\frac{\partial^2 u}{\partial \xi^2}+2B\frac{\partial^2 u}{\partial \xi\partial \eta}+C\frac{\partial^2 u}{\partial \eta^2}+D\left(\frac{\partial u}{\partial \xi},\frac{\partial u}{\partial \eta},u,\xi,\eta\right)=0$$

With J given by

$$J=\frac{\partial(\xi,\eta)}{\partial(x,y)}=\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y}-\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}$$

it can be shown that

$$B^2-AC=(b^2-ac)J^2$$

What can be deduced from the above equation? [20%]

(b) Classify the following PDEs and find changes of variables that reduce each PDE to the standard form. You are not required to reduce the PDEs to their standard form.

(i) $4\frac{\partial^2 u}{\partial x^2}+4\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 u}{\partial y^2}+x\frac{\partial u}{\partial y}=0$ [20%]

(ii) $\frac{\partial^2 u}{\partial x^2}+2\frac{\partial^2 u}{\partial x\partial y}+\sin^2(x)\frac{\partial^2 u}{\partial y^2}=0$ [20%]

(iii) $\frac{\partial^2 u}{\partial x^2}-2\sin x\frac{\partial^2 u}{\partial x\partial y}+(2-\cos^2 x)\frac{\partial^2 u}{\partial y^2}+u=0$ [20%]

(c) Reduce the equation in (b)(iii) to its standard form. [20%]

You may use

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial \xi}{\partial x}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial x}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial x}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial x^2}\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial x^2}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial \xi}{\partial x}\frac{\partial \xi}{\partial y}\frac{\partial^2 u}{\partial \xi^2} + \left(\frac{\partial \xi}{\partial x}\frac{\partial \eta}{\partial y} + \frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial x}\right)\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial \eta}{\partial x}\frac{\partial \eta}{\partial y}\frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial x \partial y}\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial x \partial y}\frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial y^2} = \left(\frac{\partial \xi}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial \xi}{\partial y}\frac{\partial \eta}{\partial y}\frac{\partial^2 u}{\partial \xi \partial \eta} + \left(\frac{\partial \eta}{\partial y}\right)^2 \frac{\partial^2 u}{\partial \eta^2} + \frac{\partial^2 \xi}{\partial y^2}\frac{\partial u}{\partial \xi} + \frac{\partial^2 \eta}{\partial y^2}\frac{\partial u}{\partial \eta}$$

4 (a) The inviscid Burgers' equation reads

$$\frac{\partial \rho}{\partial t} + \rho \frac{\partial \rho}{\partial x} = 0$$

where $\rho = \rho(x,t)$ depends on position x and time t.

(i) For the initial condition

$$\rho(x,0) = \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & x < 0 \end{cases}$$

find $\rho(x,t)$ by integrating along characteristic lines, or otherwise. [20%]

(ii) Do discontinuities form in this case? If so, indicate where and when a discontinuity first occurs and find the density distribution ρ at this instant. Otherwise, find the density distribution ρ at $t = 1$. [20%]

(b) A model equation for traffic flow reads

$$\frac{\partial \rho}{\partial t} + \frac{\partial \left((1-\rho)\rho \right)}{\partial x} = 0$$

where $\rho = \rho(x,t)$ depends on position x and time t.

(i) Consider the same initial condition as in (a)(i). By integrating along characteristic lines, or otherwise, find $\rho(x,t)$. [20%]

(ii) Do discontinuities form in this case? If so, indicate where and when a discontinuity first occurs, and find the density distribution ρ at this instant. Otherwise, find the density distribution ρ at $t = 1$. [20%]

(iii) Sketch the characteristic lines and car trajectories of the solution to (b)(i). Note: the car velocity is $u = 1 - \rho$. [20%]

END OF PAPER

Engineering Tripos Part IIA and Part IIB
Module 4M12: Partial Differential Equations and Variational Methods

Data Sheet

1 Index notation

1. $\delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases}$

2. $\epsilon_{ijk} = \begin{cases} 0 & \text{if any two of } i,j,k \text{ are equal} \\ 1 & \text{if } (i,j,k) \text{ is a permutation of (123)} \\ -1 & \text{if } (i,j,k) \text{ is a permutation of (321)} \end{cases}$

3. $[\boldsymbol{x} \times \boldsymbol{y}]_i = \epsilon_{ijk} x_j y_k$

4. $\epsilon_{ijk}\epsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}$

5. $\dfrac{\partial x_i}{\partial x_j} = \delta_{ij}$

6. $[\text{grad}\,\phi]_i = [\nabla\phi]_i = \dfrac{\partial \phi}{\partial x_i}, \quad \text{div}\,\boldsymbol{u} = \nabla \cdot \boldsymbol{u} = \dfrac{\partial u_i}{\partial x_i}, \quad [\text{curl}\,\boldsymbol{u}]_i = [\nabla \times \boldsymbol{u}]_i = \epsilon_{ijk}\dfrac{\partial u_k}{\partial x_j}$

   $[\text{grad}\,\boldsymbol{u}]_{ij} = [\nabla \boldsymbol{u}]_{ij} = \dfrac{\partial u_i}{\partial x_j}, \quad [\text{div}\boldsymbol{A}]_i = [\nabla \cdot \boldsymbol{A}]_i = \dfrac{\partial A_{ij}}{\partial x_j},$

2 Integral theorems

1. Divergence theorem:

   $$\int_V \frac{\partial}{\partial x_i}(\cdot)\, dV = \int_S (\cdot) n_i \, dS$$

   where V is a volume enclosed by the surface S, $(\cdot)$ is any permissible index notation expression, and $\boldsymbol{n}$ is the unit outward normal vector to the volume V.

2. Stokes's theorem:

   $$\int_S \epsilon_{ijk} \frac{\partial}{\partial x_j}(\cdot) n_i \, dS = \oint_C (\cdot) s_k \, dC$$

   where S is surface (possible curved) with a curve C running around the boundary of S, $\boldsymbol{n}$ is the unit normal vector to the surface S, $(\cdot)$ is any permissible index notation expression and $\boldsymbol{s}$ is the unit vector tangential to the edge of S.

3 Variational methods

1. To minimise $I = \int_0^L F(y, y', x)\, dx$, F must satisfy the Euler-Lagrange equation

$$\frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

at all x.

2. For the above problem, if F depends on y' but not on y, then

$$\frac{\partial F}{\partial y'} = k, \quad \text{where } k \text{ is a constant.}$$

If F does not depend explicitly on x (i.e., only depends on x via y and y'), then

$$F - y'\frac{\partial F}{\partial y'} = k, \quad \text{where } k \text{ is a constant.}$$

3. The directional derivative of the functional $f(\boldsymbol{u})$ is given by

$$Df(\boldsymbol{u})[\boldsymbol{v}] = \left.\frac{df(\boldsymbol{u} + \epsilon \boldsymbol{v})}{d\epsilon}\right|_{\epsilon=0}$$

4 Partial differential equations

1. Classification: The second-order quasi-linear partial differential equation

$$a\frac{\partial^2 u}{\partial x^2} + 2b\frac{\partial^2 u}{\partial x \partial y} + c\frac{\partial^2 u}{\partial y^2} + F\left(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right) = 0$$

is:

	hyperbolic	where $b^2 - ac > 0$
is:	*parabolic*	where $b^2 - ac = 0$
	elliptic	where $b^2 - ac < 0$

2. Well-posed problem: A problem is well-posed if the solution
   a) exists
   b) is unique
   c) depends continuously on the input data (i.e. is stable with respect to changes in the input data)

3. Common reference equations are:

Helmholtz equation	$\nabla^2 u + k^2 u = 0$
Poisson equation	$\nabla^2 u = f(x,y)$
Laplace equation	$\nabla^2 u = 0$
Wave equation	$\dfrac{\partial^2 u}{\partial t^2} - c^2\nabla^2 u = 0$
Diffusion equation	$\dfrac{\partial u}{\partial t} - \alpha\nabla^2 u = 0$

The form of the Laplacian operator ∇^2 in various coordinate systems can be found in the Maths Data Book.

4. D'Alembert travelling wave solution: the solution of

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial t^2} \quad \text{for } t > 0 \text{ and for all } x$$

with the initial conditions $u(x,0) = \phi(x)$ and $\partial u(x,0)/\partial t = \psi(x)$ is

$$u(x,t) = \frac{1}{2}\left(\phi(x+ct) + \phi(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(\xi)\, d\xi.$$

5. Fundamental solution (free-space Green's function):
2D Poisson/Laplace equation

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x}_0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0), \quad G(\boldsymbol{x},\boldsymbol{x}_0) = \frac{1}{2\pi}\ln|\boldsymbol{x}-\boldsymbol{x}_0|$$

3D Poisson/Laplace equation

$$\nabla^2 G(\boldsymbol{x},\boldsymbol{x}_0) = \delta(\boldsymbol{x}-\boldsymbol{x}_0), \quad G(\boldsymbol{x},\boldsymbol{x}_0) = -\frac{1}{4\pi|\boldsymbol{x}-\boldsymbol{x}_0|}$$

Fundamental solution:
Diffusion equation

$$\frac{\partial F}{\partial t} - \alpha\frac{\partial^2 F}{\partial x^2} = \delta(x-x_0)\,\delta(t-t_0)$$

$$F(x,t;x_0,t_0) = \frac{1}{\sqrt{4\alpha\pi(t-t_0)}}\exp\left(-\frac{(x-x_0)^2}{4\alpha(t-t_0)}\right) \quad \text{for } t > t_0$$

3-(space)D wave equation

$$\frac{\partial^2 F}{\partial t^2} - c^2\nabla^2 F = \delta(t-t_0)\,\delta(\boldsymbol{x}-\boldsymbol{x}_0)$$

$$F(\boldsymbol{x},t;\boldsymbol{x}_0,t_0) = \frac{\delta\left(t - t_0 - \dfrac{|\boldsymbol{x}-\boldsymbol{x}_0|}{c}\right)}{4\pi c^2|\boldsymbol{x}-\boldsymbol{x}_0|} \quad \text{for } t > t_0$$