ENGINEERING TRIPOS PART IIB
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Tuesday 10 May 2011 2.30 to 4

Module 4M13

COMPLEX ANALYSIS AND OPTIMIZATION

*Answer not more than **three** questions.*

The questions may be taken from any section.

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Answers to questions in each section should be tied together and handed in separately.

Attachment:
4M13 data sheet (4 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

gtp03

SECTION A

1 Evaluate the following integrals using contour integration. Sketch the contour that you use and show your calculation explicitly for each section of the contour.

(a) $\int_0^\infty \frac{1}{1+x^2}dx$ [40%]

(b) $\int_0^\infty \frac{\log x}{1+x^2}dx$ [30%]

(c) $\int_0^\infty \frac{(\log x)^2}{1+x^2}dx$ [30%]

2 (a) Calculate the Fourier Transform $F(\omega)$ of $f(x)$, where $f(x)$ is given by

$$f(x) = \begin{cases} 0 & x \leq 0 \\ e^{-ax} & x > 0 \end{cases}$$

where $a > 0$. [20%]

(b) Write down the formula for the inverse transform of $F(\omega)$ and evaluate the integral using contour integration in the complex ω plane. [80%]

SECTION B

3 (a) Starting from the Taylor series expansion for the value of a univariate function $f(x)$ at a point x_{k+1} near a point x_k, derive Newton's method, i.e. show that successive estimates of the location of the minimum of $f(x)$ are given by

$$x_{k+1} = x_k - \frac{f'(x_k)}{f''(x_k)}$$

where f' and f'' are the first and second derivatives of f with respect to x. [15%]

(b) An engineer is designing a cable-stayed bridge. She wants to estimate the optimal height of the towers using the simple idealization of a single section of the bridge shown in Fig. 1.

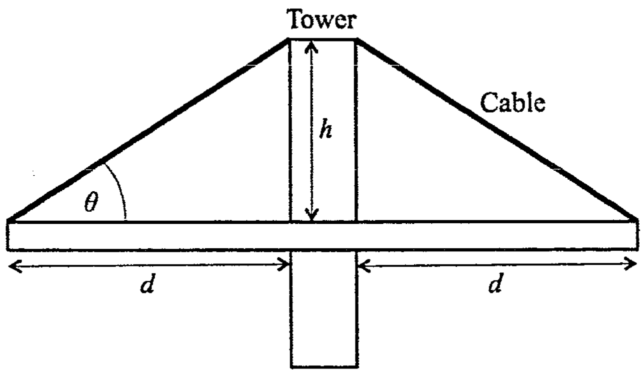


Fig. 1

To avoid failure the tower must have a cross-sectional area $A_T = W/\sigma_{0T}$, where W is the weight of the roadway and σ_{0T} is the maximum allowable stress in the tower, while the cable must have a cross-sectional area $A_C = W/(2\sigma_{0C}\sin\theta)$, where σ_{0C} is the maximum allowable stress in the cable.

The cable has a cost per unit volume C_C, while the cost per unit volume of the tower is C_T. The parameter d, which defines the length of the section, is fixed.

(i) Show that the variable cost of one section of the bridge can be written as

$$C_{\text{section}} = \frac{C_C W d}{\sigma_{0C}}(2\csc 2\theta + \alpha \tan\theta)$$

where $\alpha = \dfrac{C_T \sigma_{0C}}{C_C \sigma_{0T}}$. [20%]

(ii) For the case where $\alpha = 0.5$ estimate the optimal value of θ using Newton's method. Perform three iterations starting from an initial solution $\theta_1 = 0.5$ radians. [45%]

(iii) By using appropriate optimality criteria find an analytical expression for the optimal value of θ in terms of α. Hence find the optimal value of θ for the case where $\alpha = 0.5$ and comment on the performance of Newton's method observed in (ii). [20%]

4 An engineer has been asked to design a minimum radius, horizontal, tubular tension rod of fixed length L to transmit a defined load P. The design of the rod is shown schematically in Fig. 2. The inner radius r_i and the outer radius r_o can be varied to optimize the design.

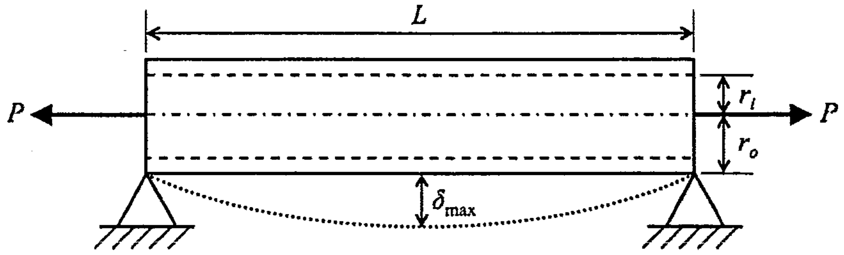


Fig. 2

The maximum load the rod can transmit without yielding is $\sigma_y \pi(r_o^2 - r_i^2)$ where σ_y is the yield stress of the material from which it is made.

The requirements specification for the rod stipulates that its maximum central deflection under self-weight $\delta_{max} = \dfrac{5\rho g L^4}{96E(r_o^2 + r_i^2)}$ must not exceed $0.001L$. Here g is the acceleration due to gravity, while ρ and E are, respectively, the density and Young's modulus of the material from which the rod is made.

(a) Show that the task of optimizing the design of the rod can be cast in the form

$$
\begin{aligned}
\text{Minimize} \quad & r_o \\
\text{Subject to} \quad & g_1 = K_1 - (r_o^2 - r_i^2) \le 0 \\
& g_2 = K_2 - (r_o^2 + r_i^2) \le 0 \\
& g_3 = r_i - r_o \le 0 \\
& g_4 = -r_i \le 0
\end{aligned}
$$

and find expressions for K_1 and K_2. [15%]

(b) For the case where $P = 50\,\text{kN}$, $L = 2.5\,\text{m}$, $\sigma_y = 200\,\text{MPa}$, $E = 200\,\text{GPa}$ and $\rho g = 77\,\text{kN}\,\text{m}^{-3}$, identify the feasible region graphically. By superimposing contours of the objective function, identify which constraints are active at the optimum, and hence find the optimal values of r_i and r_o for this design problem. [35%]

(c) Confirm your results to (b) by considering the possible solutions obtained using the Kuhn-Tucker multiplier method. There is no need to include consideration of constraints known to be inactive at the optimum. [40%]

(d) If the engineer wants to reduce the radius of the rod further by changing the material from which it is made, which constraint would it be more beneficial to try to relax? [10%]

END OF PAPER

4M13

OPTIMIZATION

DATA SHEET

1. **Taylor Series Expansion**

For one variable:

$$f(x) = f(x^*) + (x - x^*)f'(x^*) + \frac{1}{2}(x - x^*)^2 f''(x^*) + R$$

For several variables:

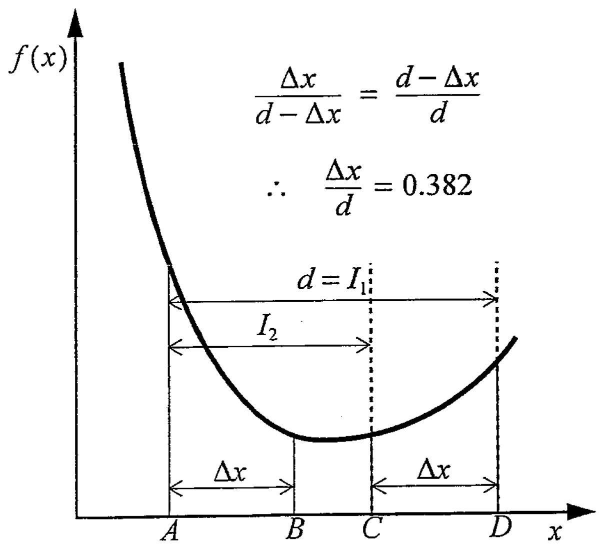
$$f(\mathbf{x}) = f(\mathbf{x}^*) + \nabla f(\mathbf{x}^*)^T (\mathbf{x} - \mathbf{x}^*) + \frac{1}{2}(\mathbf{x} - \mathbf{x}^*)^T \boldsymbol{H}(\mathbf{x}^*)(\mathbf{x} - \mathbf{x}^*) + R$$

where

$$\text{gradient } \nabla f(\mathbf{x}) = \begin{bmatrix} \frac{\partial f}{\partial x_1} \\ \vdots \\ \frac{\partial f}{\partial x_n} \end{bmatrix} \quad \text{and hessian } \boldsymbol{H}(\mathbf{x}) = \nabla(\nabla f(\mathbf{x})) = \begin{bmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_n^2} \end{bmatrix}$$

$\boldsymbol{H}(\mathbf{x}^*)$ is a symmetric $n \times n$ matrix and R includes all higher order terms.

2. **Golden Section Method**



(a) Evaluate $f(x)$ at points A, B, C and D.

(b) If $f(B) < f(C)$, new interval is $A - C$.

If $f(B) > f(C)$, new interval is $B - D$.
If $f(B) = f(C)$, new interval is either $A - C$ or $B - D$.

(c) Evaluate $f(x)$ at new interior point. If not converged, go to (b).

3. **Newton's Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -H(\mathbf{x}_k)^{-1} \nabla f(\mathbf{x}_k)$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

4. **Steepest Descent Method**

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -\nabla f(\mathbf{x}_k)$

(c) Perform line search to determine step size α_k or evaluate $\alpha_k = \dfrac{\mathbf{d}_k^T \mathbf{d}_k}{\mathbf{d}_k^T H(\mathbf{x}_k) \mathbf{d}_k}$

(d) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$

(e) Test for convergence. If not converged, go to step (b)

5. **Conjugate Gradient Method**

(a) Select starting point $\mathbf{x}_0$ and compute $\mathbf{d}_0 = -\nabla f(\mathbf{x}_0)$ and $\alpha_0 = \dfrac{\mathbf{d}_0^T \mathbf{d}_0}{\mathbf{d}_0^T H(\mathbf{x}_0)\, \mathbf{d}_0}$

(b) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$

(c) Evaluate $\nabla f(\mathbf{x}_{k+1})$ and $\beta_k = \left[\dfrac{|\nabla f(\mathbf{x}_{k+1})|}{|\nabla f(\mathbf{x}_k)|} \right]^2$

(d) Determine search direction $\mathbf{d}_{k+1} = -\nabla f(\mathbf{x}_{k+1}) + \beta_k \mathbf{d}_k$

(e) Determine step size $\alpha_{k+1} = -\dfrac{\mathbf{d}_{k+1}^T \nabla f(\mathbf{x}_{k+1})}{\mathbf{d}_{k+1}^T H(\mathbf{x}_{k+1})\, \mathbf{d}_{k+1}}$

(f) Test for convergence. If not converged, go to step (b)

6. **Gauss-Newton Method (for Nonlinear Least Squares)**

If the minimum squared error of residuals $\mathbf{r}(\mathbf{x})$ is sought:

$$\text{Minimise } f(\mathbf{x}) = \sum_{i=1}^{m} r_i^2(\mathbf{x}) = \mathbf{r}(\mathbf{x})^T \mathbf{r}(\mathbf{x})$$

(a) Select starting point $\mathbf{x}_0$

(b) Determine search direction $\mathbf{d}_k = -[\, J(\mathbf{x}_k)^T J(\mathbf{x}_k)\,]^{-1} J(\mathbf{x}_k)^T \mathbf{r}(\mathbf{x}_k)$

where $J(\mathbf{x}) = \begin{bmatrix} \nabla r_1(\mathbf{x})^T \\ \vdots \\ \nabla r_m(\mathbf{x})^T \end{bmatrix} = \begin{bmatrix} \frac{\partial r_1}{\partial x_1} & \cdots & \frac{\partial r_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial r_m}{\partial x_1} & \cdots & \frac{\partial r_m}{\partial x_n} \end{bmatrix}$

(c) Determine new estimate $\mathbf{x}_{k+1} = \mathbf{x}_k + \mathbf{d}_k$

(d) Test for convergence. If not converged, go to step (b)

7. **Lagrange Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda} = [\lambda_1, \ldots, \lambda_m]^T$ is the vector of Lagrange multipliers and

$$[\nabla \mathbf{h}(\mathbf{x}^*)]^T = \begin{bmatrix} \nabla h_1(\mathbf{x}^*) & \ldots & \nabla h_m(\mathbf{x}^*) \end{bmatrix} = \begin{bmatrix} \frac{\partial h_1}{\partial x_1} & \cdots & \frac{\partial h_m}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \frac{\partial h_1}{\partial x_n} & \cdots & \frac{\partial h_m}{\partial x_n} \end{bmatrix}$$

8. **Kuhn-Tucker Multipliers**

To minimise $f(\mathbf{x})$ subject to m equality constraints $h_i(\mathbf{x}) = 0, i = 1, \ldots, m$ and p inequality constraints $g_i(\mathbf{x}) \le 0, i = 1, \ldots, p$, solve the system of simultaneous equations

$$\begin{aligned} \nabla f(\mathbf{x}^*) + [\nabla \mathbf{h}(\mathbf{x}^*)]^T \boldsymbol{\lambda} + [\nabla \mathbf{g}(\mathbf{x}^*)]^T \boldsymbol{\mu} &= 0 \quad (n \text{ equations}) \\ \mathbf{h}(\mathbf{x}^*) &= 0 \quad (m \text{ equations}) \\ \forall\ i = 1, \ldots, p, \quad \mu_i g_i(\mathbf{x}) &= 0 \quad (p \text{ equations}) \end{aligned}$$

where $\boldsymbol{\lambda}$ are Lagrange multipliers and $\boldsymbol{\mu} \ge 0$ are the Kuhn-Tucker multipliers.

9. **Penalty & Barrier Functions**

To minimise $f(\mathbf{x})$ subject to p inequality constraints $g_i(\mathbf{x}) \le 0$, $i = 1, \ldots, p$, define

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) + p_k P(\mathbf{x})$$

where $P(\mathbf{x})$ is a penalty function, e.g.

$$P(\mathbf{x}) = \sum_{i=1}^{p} (\max[0,\ g_i(\mathbf{x})])^2$$

or alternatively

$$q(\mathbf{x}, p_k) = f(\mathbf{x}) - \frac{1}{p_k} B(\mathbf{x})$$

where $B(\mathbf{x})$ is a barrier function, e.g.

$$B(\mathbf{x}) = \sum_{i=1}^{p} \frac{1}{g_i(\mathbf{x})}$$

Then for successive $k = 1, 2, \ldots$ and p_k such that $p_k > 0$ and $p_{k+1} > p_k$, solve the problem

minimise $q(\mathbf{x}, p_k)$