ENGINEERING TRIPOS PART IIB 2012
4A10 FLOW INSTABILITY

Dr M P Juniper
Dr A Agarwal

1(a) $p' = \gamma \nabla \cdot \underline{n} = \pm \gamma \left(-\frac{\partial^2 \zeta'}{\partial x^2} - \frac{\partial^2 \zeta'}{\partial y^2} \right)$ on $z = \pm a$ — (1)

but $w' = \frac{\partial \zeta'}{\partial t}$ on $z = \pm a$ — (2)

& from the z-momentum equation,

$$\rho \frac{\partial w'}{\partial t} = -\frac{\partial p'}{\partial z} \quad -(3)$$

$\therefore$ From Eq (2),

$$\rho \frac{\partial^2 \zeta'}{\partial t^2} = -\frac{\partial p'}{\partial z}$$

Applying $\rho \frac{\partial^2}{\partial t^2}$ to Eq (1):

$$\boxed{\rho \frac{\partial^2 p'}{\partial t^2} = \pm \gamma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial p'}{\partial z}} \text{ on } z = \pm a \quad -(4)$$

(b) $p' = \cancel{\ } f(z)\, e^{st + i(kx + ly)}$

substituting in $\nabla^2 p' = 0$

gives

$$\frac{d^2 f}{dz^2} - (k^2 + l^2) f = 0$$

$$\Rightarrow f = A \cosh \sqrt{k^2+l^2}\, z + B \sinh \sqrt{k^2+l^2}\, z \quad -(5)$$

(c) For the symmetric mode, $B = 0$.

Using B.C. eq(4) on $z = a$:

$$\rho s^2 \cosh \sqrt{k^2+l^2}\, a = \pm \gamma (-k^2 - l^2) \sqrt{k^2+l^2} \sinh \sqrt{k^2+l^2}\, a \quad -(6)$$

$$\therefore \rho s^2 = -\frac{\gamma \tilde{\alpha}^3}{a^3} \tanh \tilde{\alpha}$$

where $\tilde{\alpha} = a \sqrt{k^2+l^2}$

We get the same result if we apply the B-C at $z = -a$.

$$\therefore \quad \boxed{\frac{a^3 \rho s^2}{\gamma} = -\tilde{\alpha}^3 \tanh \tilde{\alpha}} \qquad -(7)$$

For the odd mode $A = 0$. Then, Eq (6) is replaced by:

$$\rho s^2 \sinh(\sqrt{k^2+\ell^2}\,a) = \gamma(-k^2-\ell^2)\sqrt{k^2+\ell^2}\cosh\sqrt{k^2+\ell^2}\,a$$

$$\Rightarrow \quad \boxed{\frac{a^3 \rho s^2}{\gamma} = -\tilde{\alpha}^3 \coth \tilde{\alpha}} \qquad -(8)$$

(d) We find from Eqs (7) and (8) that s^2 is always negative. This is because $\tilde{\alpha}$, a, ρ & γ are all positive.

$\Rightarrow$ s is purely imaginary. Therefore, the plane jet in this problem is always stable.

Examiner's note:

(a) This section was answered correctly by only 10% of the students. The subsequent sections were designed to be independent, so this did not affect their performance on the other sections.
(b) This section was answered well by almost all students.
(c) This section was answered well by most students.
(d) This section was answered correctly by about half the students. Some students did not know the correct criterion for instability and some did not know that tanh and coth functions are always positive for a positive argument.

2(a)(i) Suppose a fluid particle is raised by a small height dz. If $T_0 > T_1$, this particle is hotter and therefore less dense than its surroundings. Hence, the upward buoyancy force exceeds the weight of the particle and so it continues to move upwards. Hence the system is unstable.

(ii) Rayleigh number. It is the ratio of buoyancy forces to thermo-viscous forces.

— Turn over for 2(b)

$*$ 2(b) $$\frac{dw}{dt} = \frac{-\nu}{d^2} w + \alpha g (\theta - \bar{\theta})$$

$$\frac{d\theta}{dt} = -\frac{\kappa}{d^2} (\theta - \bar{\theta}) + \frac{T_0 - T_1}{d} w$$

(i) When $w = 0$ & $\theta = \bar{\theta}$

$$\frac{dw}{dt} = \frac{d\theta}{dt} = 0 \Rightarrow \text{equilibrium.}$$

(ii) Let $w = w'$

$\theta = \bar{\theta} + \theta'$, Let $\left(\frac{T_0 - T_1}{d}\right) = \beta$

$$\frac{dw'}{dt} = \frac{-\nu}{d^2} w' + \alpha g \theta'$$

$$\frac{d\theta'}{dt} = -\frac{\kappa}{d^2} \theta' + \beta w'$$

Let $w' = \hat{w} e^{st}$

$\theta' = \hat{\theta} e^{st}$

$$\therefore s\hat{w} = -\frac{\nu}{d^2} \hat{w} + \alpha g \hat{\theta}$$

$$s\hat{\theta} = \frac{-\kappa}{d^2} \hat{\theta} + \beta \hat{w}$$

$$\therefore s\hat{w} = \frac{-\nu}{d^2} \hat{w} + \alpha g \left\{ \frac{\beta \hat{w}}{\left(s + \frac{\kappa}{d^2}\right)} \right\}$$

$$\Rightarrow s + \frac{\nu}{d^2} - \frac{\alpha g \beta}{s + \frac{\kappa}{d^2}} = 0$$

$$\Rightarrow \quad s^2 + \left(\frac{\kappa}{d^2} + \frac{\nu}{d^2}\right)s + \frac{\kappa\nu}{d^4} - \alpha g\beta = 0.$$

$$s = \frac{1}{2}\left\{-\frac{(\kappa+\nu)}{d^2} \pm \sqrt{\frac{(\kappa+\nu)^2}{d^4} - 4\left(\frac{\kappa\nu}{d^4} - \alpha g\beta\right)}\right\}$$

$$s = \frac{1}{2}\left\{-\frac{\kappa+\nu}{d^2} \pm \sqrt{\frac{(\kappa-\nu)^2}{d^4} + 4\alpha g\beta}\right\}$$

~~Get~~ Unstable when $s > 0$

$$\Rightarrow \left(\sqrt{\frac{(\kappa-\nu)^2}{d^4} + 4\alpha g\beta}\right)^2 > \frac{(\kappa+\nu)^2}{d^4}$$

i.e. $$\frac{(\kappa-\nu)^2}{d^4} - \frac{(\kappa+\nu)^2}{d^4} + 4\alpha g\beta > 0.$$

or $$-\frac{\cancel{4}\kappa\nu}{d^4} + \cancel{4}\alpha g\beta > 0$$

or $$\frac{\alpha g\beta d^4}{\kappa\nu} > 1$$

or $$\boxed{\alpha g \frac{(T_0 - T_1)}{d} \frac{d^4}{\kappa\nu} > 1}$$

* This question is from the book, Introduction to hydrodynamic Stability by P.G Drazin. Cambridge University Press, 2002

Examiner's note Q2:

(a) This section was answered well by almost all students
(b) The first part of this section was answered well by most students. Some students did not understand that it is not sufficient to have a steady velocity for equilibrium. They did not show that temperature should be steady as well. The final part of this section can be solved in a couple of different ways. Most students did well on this part.

3.
a) a_1 represents advection. Typically this would be the mean flow velocity.
a_2 represents diffusion. Typically this would be the kinematic viscosity.
[This question was well answered]

b) Substituting $u = u_0 \exp(i(kx - \omega t))$ into $\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} = a_2 \frac{\partial^2 u}{\partial x^2}$ leads to:

$$-i\omega u + a_1 i k u = -a_2 k^2 u$$

$$\Rightarrow u_0 = 0 \quad \text{or} \quad \omega = a_1 k - i a_2 k^2$$

i) $\Rightarrow \quad \frac{\omega}{k} = a_1 - i a_2 k$; this is the phase velocity

ii) $\Rightarrow \quad \frac{d\omega}{dk} = a_1 - 2 i a_2 k$; this is the group velocity

iii) The phase velocity differs from the group velocity so this system is dispersive. An equivalent argument is to note that ω/k depends on k, so different wavenumbers travel at different wave speeds, so the system is dispersive.
[This section was well answered]

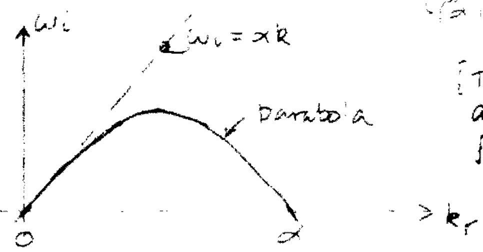
c) The perturbation is temporally unstable for any real k that has $\omega_i > 0$

$$\omega = a_1 k - i a_2 k^2 \quad \text{where} \quad a_1 = -i + \alpha i, \quad a_2 = 1, \quad \alpha \in \mathbb{R}$$
(α is a real number)

$$= (\alpha i - i)k - i k^2$$

i) k is real so $\omega_i = (\alpha k - k^2) i$
$= k(\alpha - k) i$

$\therefore$ temporally unstable for $0 < k < \alpha$.



[This was well answered, apart from the graph sketch]

ii) $\frac{d\omega}{dk} = 0$ when $a_1 = 2 i a_2 k \quad \Rightarrow \quad k = \frac{a_1}{2 i a_2} = \frac{\alpha i - 1}{2i} = \frac{\alpha}{2} + \frac{i}{2}$

[This section was very well answered]

iii) The flow is absolutely unstable if the waves with zero group velocity have positive growth rate, i.e. $\omega_i > 0$ for waves with $d\omega/dk = 0$
[Around 25% of students thought that the flow was A.U. if $d\omega/dk > 0$]

$$\omega = a_1 k - i a_2 k^2, \text{ and } a_1 = 2 i a_2 k \text{ when } d\omega/dk = 0$$

$\Rightarrow \quad \omega = 2 i a_2 k^2 - i a_2 k^2 = i a_2 k^2$; but $a_2 = 1$ and $k = \frac{\alpha + i}{2}$ when $\frac{d\omega}{dk} = 0$

$$\Rightarrow \quad \omega = \frac{i}{4}(\alpha + i)^2 = \frac{i}{4}(\alpha^2 + 2i\alpha - 1)$$

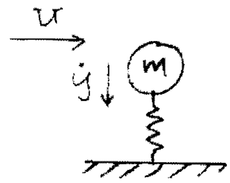
This has positive imaginary component ($\omega_i > 0$) when $\alpha^2 > 1$.
Therefore the flow is unstable when $\alpha^2 > 1$.
[This was well answered by the students who had the correct definition of absolute instability]

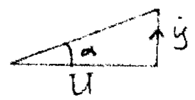
M P JUNIPER, May 2012

Examiner's note Q3:

(a) This section was answered well by almost all candidates
(b) This section was answered well by almost all candidates, although around 25% of candidates did not give the correct definition of a dispersive medium.
(c) The algebra in this section was done well but few candidates could sketch the graph, which was suprising.
(d) Around 25% of candidates thought that the flow is absolutely unstable if the group velocity is positive, so clearly have not grasped the concept of absolute instability. The majority answered this well, however,

4. (a)





α = apparent angle of attack in the frame of reference moving with the body.

$\alpha = \tan^{-1}\left(\dot{y}/U\right)$

The first term in the Taylor expansion around $\dot{y}/U = 0$ is $\alpha = \dot{y}/U$

so $\alpha \approx \dot{y}/U$ is valid when $\dot{y}/U \ll 1$.

By substitution, $C_y = \dfrac{\dot{y}}{U} - r\left(\dfrac{\dot{y}}{U}\right)^3$. (This is required later)

(b) total energy = Kinetic energy + potential energy in the spring

$$E = \tfrac{1}{2}m(\dot{y})^2 + \tfrac{1}{2}k(y)^2$$

$$\Rightarrow \frac{dE}{dt} = \tfrac{1}{2}m\left(2\dot{y}\ddot{y}\right) + \tfrac{1}{2}k\left(2y\dot{y}\right) = m\dot{y}\ddot{y} + ky\dot{y} = (m\ddot{y} + ky)\dot{y}$$

But we know that $(m\ddot{y} + ky) = qU^2 C_y - p\dot{y} = qU\dot{y} - qr\dfrac{\dot{y}^3}{U} - p\dot{y}$

$$\Rightarrow \frac{dE}{dt} = (qU - p)\dot{y}^2 - \frac{qr}{U}\dot{y}^4$$

$\dot{y}^2$ and $\dot{y}^4$ are always positive. Therefore the first term causes an increase in energy over one period if $(qU > p)$ and a decrease if $(p > qU)$.
The second term is always negative because q, r, and U are all positive.
[Both terms oscillate harmonically, but the most important point to note was their sign].

(c) $y = a\cos(\omega t) \Rightarrow \dot{y} = -\omega a\sin(\omega t)$ where a is the amplitude

$$\Rightarrow \int_0^{2\pi/\omega} \frac{dE}{dt}\,dt = \int_0^{2\pi/\omega} (qU - p)(\omega a)^2 \sin^2(\omega t)\,dt - \int_0^{2\pi/\omega} \frac{qr}{U}(\omega a)^4 \sin^4(\omega t)\,dt$$

$$= (qU - p)(\omega a)^2\left(\frac{\pi}{\omega}\right) - \frac{qr}{U}(\omega a)^4\left(\frac{3\pi}{4\omega}\right)$$

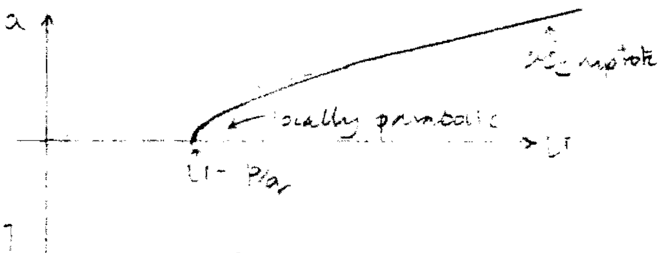
If this integral is positive then the amplitude of the system will grow (on a long timescale)
When the system first becomes unstable, a is infinitesimal, so the second term is negligible compared with the first term. ω is positive so the amplitude will grow when $(qU - p) > 0$. [This result can be derived by linearizing the governing equation about $y = 0$ but then one does not have the second term in the above equation, which makes part (d) impossible].

(d) When a is constant, $\displaystyle\int_0^{2\pi/\omega} \frac{dE}{dt}\,dt = 0 \Rightarrow (qU - p)(\omega a)^2\left(\frac{\pi}{\omega}\right) = \frac{qr}{U}(\omega a)^4\left(\frac{3\pi}{4\omega}\right)$

$$\Rightarrow a = 0 \text{ or } a^2 = \frac{U(qU - p)}{qr\omega^2}\frac{4}{3}$$

- This has real solutions for $U \geq p/q$
- Around $U = p/q$ it is parabolic
- As $U \to \infty$, a is proportional to U

[The standard form of the graph is $a^2 = U(U-1)$]



M P Juniper Mar 2017

Examiner's note Q4:

This question requires the candidates to derive a familiar result in an unfamiliar way. This seemed to confuse around half the candidates and there were several low marks in this question.

(a) This section was well answered.
(b) Around half the candidates followed the technique suggested in the question and wrote down the energy as the sum of kinetic energy and the potential energy in the spring. Several tried alternative methods, some of which were successful, but around 30% of candidates were thrown by this relatively simple section. Very few candidates commented on how these terms vary over one cycle.
(c) Most of the candidates who reached this section managed to perform the algebra correctly, but few followed the instruction to explain their reasoning.
(d) Again, most candidates performed the algebra correctly but few commented on the dependence and even fewer managed to sketch the graph correctly.