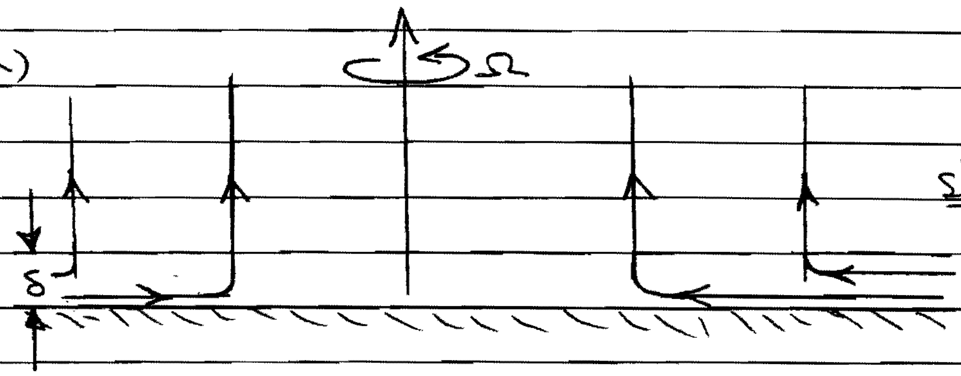
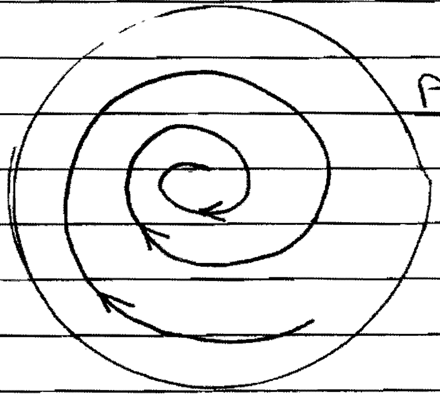
ENGINEERING TRIPOS PART IIB 2012
4A12 TURBULENCE AND VORTEX DYNAMICS **Prof P A Davidson**

CRIB 2012

1 (a)



Side View



Plan View

Outside the boundary layer we have the balance

$$\frac{\partial p}{\partial r} = \rho \Omega^2 r$$

This sets up a radial pressure gradient which is imposed on the boundary layer.

Within the boundary layer the radial pressure forces are inward and exceed the centrifugal force, and so drive fluid radially inward.

(b) (i) In boundary layer $\frac{\partial p}{\partial r} \sim \rho \nu \frac{\partial^2 u_r}{\partial z^2}$

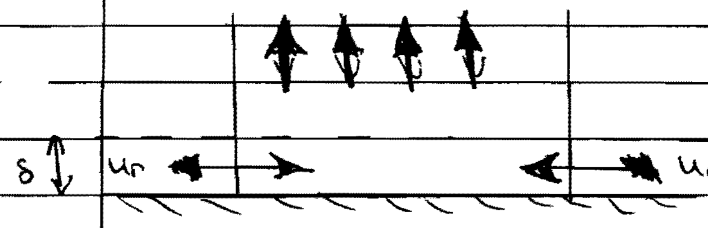
But $\frac{\partial p}{\partial r} \sim \rho \Omega^2 r \sim \rho \frac{u_\theta^2}{r}$, $\frac{\partial^2 u_r}{\partial z^2} \sim \frac{u_r}{\delta^2}$

$$\Rightarrow \underline{\underline{\rho \frac{u_\theta^2}{r} \sim \rho \nu \frac{u_r}{\delta^2}}}$$

(ii) $u_r \sim u_\theta \sim \Omega r \Rightarrow \Omega \sim \frac{\nu}{\delta^2} \Rightarrow \underline{\underline{\delta \sim \sqrt{\frac{\nu}{\Omega}}}}$

(iii) Apply continuity to a cylinder of radius r

$$|u_z| \pi r^2 \sim u_r 2\pi r \delta$$



$$|u_z| \sim \frac{u_r\delta}{r} \sim \frac{u_\theta\delta}{r} \sim \Omega\delta$$

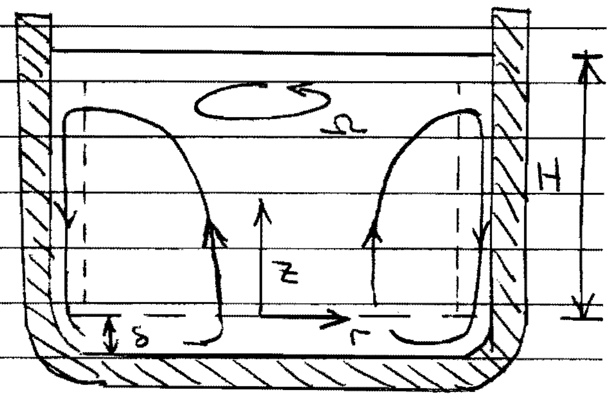
$$\Rightarrow \quad |u_z| \sim \sqrt{\nu\Omega}\,.$$

(iv) There is no imposed length scale so $\delta = f(\nu, \Omega)$ and $|u_z| = g(\nu, \Omega)$. Consider $\delta = f(\nu,\Omega)$: 3 variables and 2 dimensions $\Rightarrow$ 1 dimensionless group, i.e. $\dfrac{\delta}{\sqrt{\nu/\Omega}}$.

$\Rightarrow \delta \sim \sqrt{\nu/\Omega}$. Similarly for $|u_z| = g(\nu, \Omega)$.

(c)

(i) continuity tells us radial outflow in core equals radial inflow in Bodewadt layer.



$$u_r^{(CORE)} H \sim u_r^{(B.L.)}\delta$$

But $u_r^{(B.L.)} \sim u_\theta^{(B.L.)} \sim \Omega r \Rightarrow u_r^{(CORE)} H \sim \Omega r \delta$

$$\Rightarrow u_r^{(CORE)} H \sim \sqrt{\nu\Omega}\, r$$

In core: $$\frac{\partial u_z}{\partial z} = -\frac{1}{r}\frac{\partial}{\partial r}(r u_r) \sim -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\sqrt{\nu\Omega}}{H} r^2\right) \sim -\frac{\sqrt{\nu\Omega}}{H}$$

$$\Rightarrow u_z^{(CORE)} \sim \frac{\sqrt{\nu\Omega}}{H}(\text{constant} - z) \Rightarrow u_z^{(CORE)} \sim \sqrt{\nu\Omega}\left(1 - \frac{z}{H}\right.$$

($u_z = 0$ at $z = H$)

(ii) The spin-down time is time taken to flush content of cup through boundary layer: $\tau \sim \dfrac{H}{u_z^{(CORE)}} \sim \dfrac{H}{\sqrt{\nu\Omega}}$

(3)

(2)

(a) $\omega = \frac{1}{r}\frac{\partial}{\partial r}(r u_\theta)$ (Data sheet)

$= \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\Gamma_0}{2\pi}\left(1 - e^{-r^2/\delta^2}\right)\right)$

$= -\frac{\Gamma_0}{2\pi r}\frac{\partial}{\partial r}\left(e^{-r^2/\delta^2}\right)$

$= -\frac{\Gamma_0}{2\pi \cancel{r}}\left(\frac{-2\cancel{r}}{\delta^2}\right) e^{-r^2/\delta^2}$

$= \underline{\underline{\frac{\Gamma_0}{\pi\delta^2} e^{-r^2/\delta^2}}}$

$\frac{D\omega}{Dt} = \frac{\partial\omega}{\partial t} + (\underset{\sim}{u}\cdot\nabla)\omega = \frac{\partial\omega}{\partial t}$

$\Rightarrow \frac{D\omega}{Dt} = -\frac{\Gamma_0}{\pi\delta^4}\frac{d\delta^2}{dt} e^{-r^2/\delta^2} + \frac{\Gamma_0}{\pi\delta^2}\left(\frac{r^2}{\delta^4}\frac{d\delta^2}{dt}\right)e^{-r^2/\delta^2}$

$= -\frac{\Gamma_0}{\pi\delta^4}\frac{d\delta^2}{dt} e^{-r^2/\delta^2}\left[1 - \frac{r^2}{\delta^2}\right]$

$= -\frac{\Gamma_0}{\pi\delta^4} c^2\nu\, e^{-r^2/\delta^2}\left[1 - \frac{r^2}{\delta^2}\right]$

$\nu\frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial\omega}{\partial r}\right] = \frac{\nu}{r}\frac{\partial}{\partial r}\left[r\frac{\Gamma_0}{\pi\delta^2}\left(\frac{-2r}{\delta^2}\right)e^{-r^2/\delta^2}\right]$

$= -\frac{2\nu\Gamma_0}{\pi r\delta^4}\frac{\partial}{\partial r}\left[r^2 e^{-r^2/\delta^2}\right]$

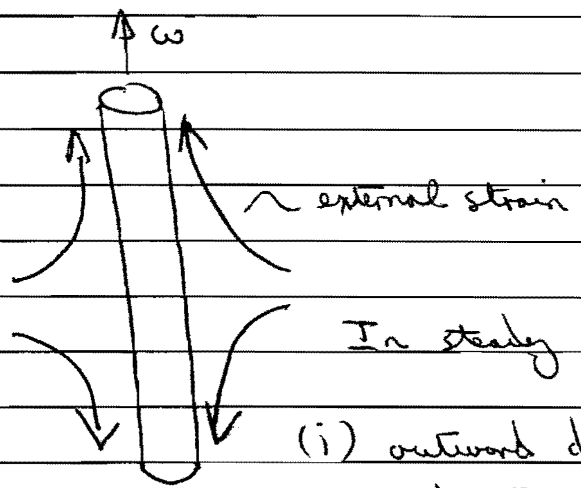
$= -\frac{2\nu\Gamma_0}{\pi r\delta^4}\left[2r e^{-r^2/\delta^2} - \frac{2r^3}{\delta^2}e^{-r^2/\delta^2}\right]$

$= -\frac{4\nu\Gamma_0}{\pi\delta^4}\left[1 - \frac{r^2}{\delta^2}\right]e^{-r^2/\delta^2}$

L.H.S. = R.H.S. provided

$c^2 = 4 \Rightarrow \underline{\underline{c = 2}}$

(b)



In steady state there is a balance of :

(i) outward diffusion of ω and inward advection of ω

(ii) diffusive reduction in ω and intensification of ω by vortex-line stretching

If $\alpha > 4\nu/\delta^2$ inward ~~diffu~~ advection dominates ~~with~~ over outward diffusion and the vortex shrinks back to $\alpha = 4\nu/\delta^2$.

If $\alpha < 4\nu/\delta^2$ outward diffusion of ω dominates over inward advection and the vortex diffuses out until we reach $\alpha = 4\nu/\delta^2$

The Burger's vortex has the property that, for $Re \to \infty$, the dissipation of energy is independent of ν, which is also ~~true~~ true in turbulence. The small scales in a turbulent flow are thought to be like Burger's vortex.

Q3

(a) The energy of the turbulence is extracted from the mean flow and feeds the large eddies. These then break down to many smaller eddies and so on. During this process, there is no energy dissipation, as the Re associated with the eddies of diminishing size stays high. At the smallest scales, the Re is not high anymore and dissipation occurs. But the dissipation rate stays constant across the eddy range. This is the concept of the energy cascade.

If ε is constant, it must be of order

$$\frac{(\text{energy to be dissipated})}{(\text{large eddy lifetime})} \simeq A \frac{k}{L/\sqrt{k}} = A \frac{k^{3/2}}{L}$$

with the constant $A = o(1)$.
(Note: we use $u^2 = k$ ignoring the $\frac{3}{2}$ factor for simplicity)

(b) If ν changes to 2ν, and if L & k are not affected, then ε does not change. What changes is the scale at which ε occurs.

Using $\varepsilon = 15\nu \frac{u^2}{\lambda^2}$ $\Rightarrow \left(\frac{\lambda_{hot}}{\lambda_{old}}\right)^2 = \left(\frac{\nu_{hot}}{\nu_{old}}\right)$

$\Rightarrow \lambda$ increases by a factor $\sqrt{2}$

Using $\eta_k = (\nu^3/\varepsilon)^{1/4}$

$\tau_k = (\nu/\varepsilon)^{1/2}$

$v_k = (\nu\varepsilon)^{1/4}$

we get that η_k increases by a factor $2^{3/4} \approx 1.68$

τ_k " " " $2^{1/2} \approx 1.41$

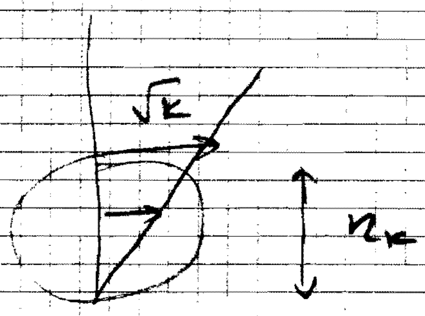
v_k " " " $2^{1/4} \approx 1.19$

Note: the fact that the small scales increase as ν increases has profound effects on how flames (and their chemistry) responds to turbulence.

(c) The k-eqtn in the case of homogeneous flow (all $\frac{\partial ( )}{\partial x_i}$ terms $\to 0$) implies no convection, turbulent diffusion, and production.

So, by necessity, $\frac{dk}{dt} = -\varepsilon$. Therefore the concept of non-decaying stationary turbulence $\left(\frac{dk}{dt} = 0\right)$ is not physically realizable. But we use it for demonstration purposes.

(d)



The concept is as above: the velocity varies by $\sqrt{k}$ across an eddy of size η_k. The dissipation associated with this is of order $\nu\left(\frac{\sqrt{k}}{\eta_k}\right)^2 = \frac{\nu k}{\eta_k^2}$

If the volume fraction of these structures is X, then the mean dissipation is

$\nu \frac{k}{\eta_k^2} X$. But this must be equal to $\frac{k\sqrt{k}}{L}$

$$\Rightarrow \quad X \frac{\nu k}{\eta_k^2} = \frac{k\sqrt{k}}{L}$$

Using $Re = \frac{\sqrt{k}L}{\nu}$ and $\eta_k = L\,Re^{-3/4}$

we get $X \frac{\nu \cancel{k}}{L^{\cancel{2}}\, Re^{-3/2}} = \frac{\cancel{k}\sqrt{k}}{\cancel{L}}$

$$\Rightarrow \quad X = \left(\frac{\sqrt{k}L}{\nu}\right) Re^{-3/2}$$

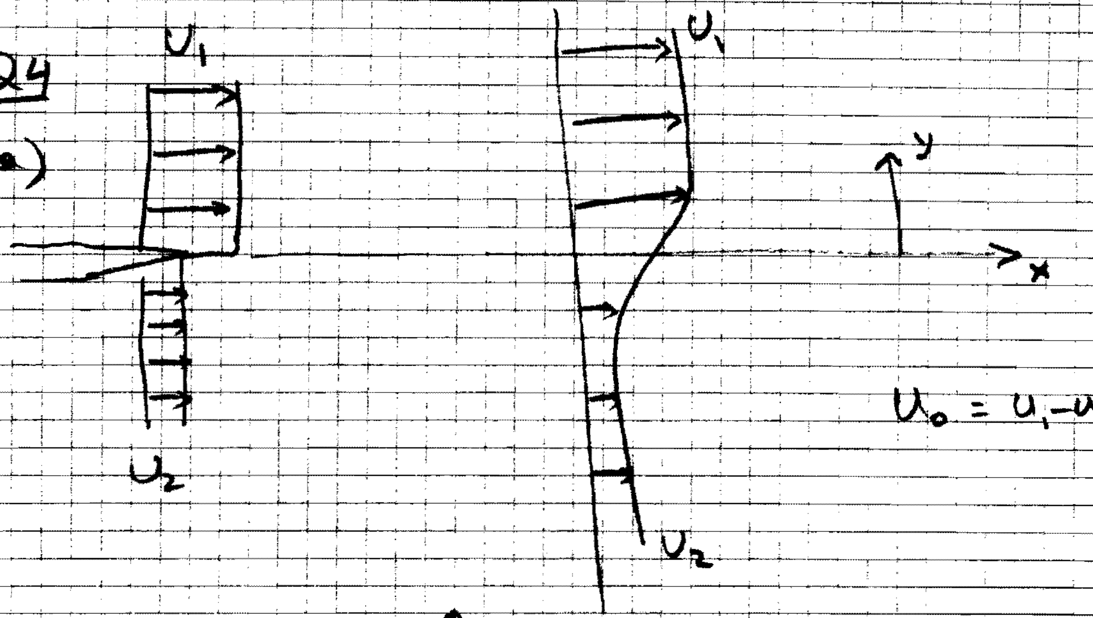
$$= \cancel{Re}$$

$$X = Re^{-1/2}$$

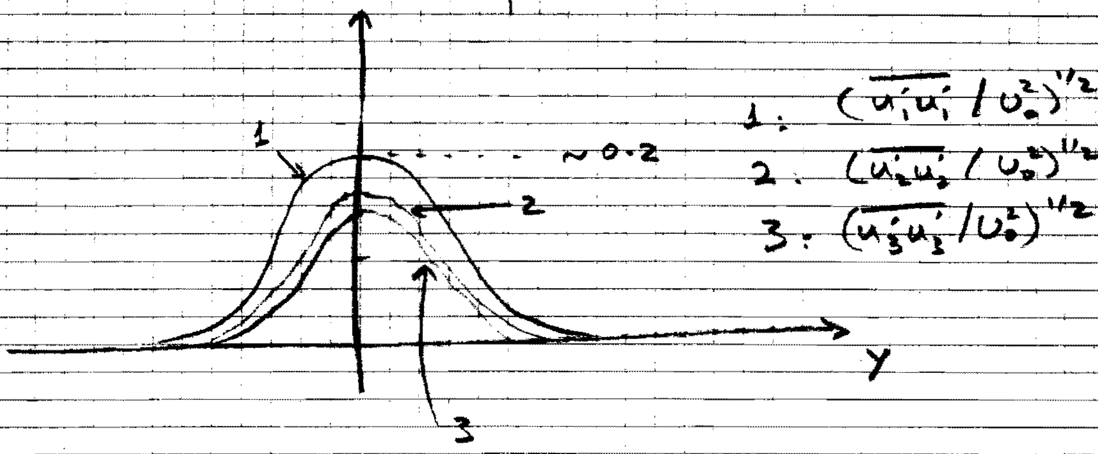
The volume fraction of the small scales ↓ as Re ↑.
Other, more refined, estimates exist in turbulence theory giving different Re scaling.

Q4

(a)



$U_0 = U_1 - U_2$



u_1 is generated by the mean shear. Not much production of the other components; they are fed by pressure interaction terms. Therefore there is anisotropy ($u_1 > u_2 \approx u_3$).

(b) At $y=0$, $\bar{u} = \dfrac{U_1 + U_2}{2} = 30$ m/s

If $u'/U_0 \sim 0.2$, $U_0 = U_1 - U_2 = 40$ m/s

$\Rightarrow u' \approx 8$ m/s

Integral lengthscale $L \simeq 0.2\,\delta \sim 10$ mm

$\Rightarrow$ eddy turnover time $= \frac{0.01}{8} \simeq 1.25$ ms

But integral time scale $\simeq \frac{L}{\bar{u}} \simeq 0.33$ ms.

(c) Self-similarity (full preservation) implies:

$\bar{u} = U_o(x)\, F(\eta)$ $\quad \eta = \frac{y}{\delta(x)} \Rightarrow \frac{d\eta}{dx} = -\frac{y}{\delta^2}\frac{d\delta}{dx}$

$\bar{v} = U_o(x)\, G(\eta)$ $\quad \Rightarrow \frac{d\eta}{dx} = -\eta \frac{1}{\delta}\frac{d\delta}{dx}$

etc for the Reynolds stresses.

For the mixing layer, $U_o = U_1 - U_2 \Rightarrow$ independent of x.
Substituting self-similar expressions in continuity:

$$\frac{d}{dx}\left(U_o F(\eta)\right) + \frac{d}{dy}\left(U_o G(\eta)\right) = 0$$

$$\Longrightarrow U_o F' \frac{d\eta}{dx} + U_o G' \frac{1}{\delta} = 0$$

$$\Longrightarrow -\eta F' \cancel{\frac{1}{\delta}} \frac{d\delta}{dx} + G' \cancel{\frac{1}{\delta}} = 0$$

$\frac{d\delta}{dx}$ must be independent of x, i.e. $\frac{d\delta}{dx} = K$

$\therefore \boxed{G' = K \eta F'}$ $\quad \Rightarrow \underline{\underline{\delta \sim x}}$

Assuming $\overline{u'v'} = -\nu_t \frac{\partial \bar{u}}{\partial y}$ and that $\nu_t = C\, U_o \delta(x)$

we get $\overline{u'v'} = -C U_o \delta\, U_o F' \frac{1}{\delta} = -C U_o^2 F'$

Substituting in streamwise momentum eqtn.

$$\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}$$

$$U_o F \left(-\frac{1}{\delta}\frac{d\delta}{dx}\right)\eta U_o F' + (U_o G)(U_o F' \frac{1}{\delta}) = C U_o^2 F'' \frac{1}{\delta}$$

$$\Rightarrow \boxed{-K\eta F F' + G F' = C F''}$$

Experiment gives K & C & numerical solution gives F & G

Examiner's note to Q3:

The first three parts were straightforward and were reasonably well answered. The final part, although a minor build-up from concepts used thoroughly in the lectures, proved very difficult with only a handful of students getting anywhere close.

Examiner's note to Q4:

The first two parts needed qualitative answers and order-of-magnitude estimates and were reasonably well answered, although few students could make reasonable estimates of the turbulent intensity. Part (c) was not completed by anyone, although it closely followed a procedure from the lecture notes.