ENGINEERING TRIPOS PART IIB 2012
4A15 AEROACOUSTICS

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1(a) From the datacard,

$$\rho_0 \frac{\partial v'}{\partial t} = -\frac{\partial p'}{\partial r}$$

Let $v' \propto e^{i\omega t}$

$$\therefore \quad i\omega\rho_0 v' = -\frac{\partial p'}{\partial r}$$

or $$v' = \frac{i}{\omega\rho_0}\frac{\partial p'}{\partial r}$$

$$\therefore \quad v_i' = \frac{i}{\omega\rho_1}\left(-\frac{i\omega}{c_1} - \frac{1}{r}\right)p_i'$$

$$= \frac{1}{\rho_1 c_1}\left(1 - \frac{ic_1}{\omega r}\right)p_i'$$

$$\boxed{v_i' = \frac{\left(1 - \frac{ic_1}{\omega r}\right)}{z_1}p_i'}$$

Similarly

$$\boxed{v_t' = \frac{\left(1 - \frac{ic_2}{\omega r}\right)}{z_2}p_t'}$$

$$v_r' = \frac{i}{\omega\rho_1}\left(\frac{i\omega}{c_1} - \frac{1}{r}\right)p_r'$$

$$\boxed{v_r' = -\frac{\left(1 + \frac{ic_1}{\omega r}\right)}{z_1}p_r'}$$

[20%]

(b) Boundary conditions at the surface $(r=a)$ are:

$$p_i' + p_r' = p_t' \quad \text{and}$$

$$v_i' + v_r' = v_t'$$

$$\therefore \quad A + B = C$$

$$\& \quad \frac{A}{Z_1}(1 - i\alpha_1) - \frac{B(1 + i\alpha_1)}{Z_1} = \frac{\cancel{C}\,(A+B)\,(1 - i\alpha_2)}{Z_2} \qquad -(1)$$

The reflection coefficient is defined as

$$R = \left.\frac{p_r}{p_i}\right|_{r=a}$$

$$= \frac{B}{A}$$

$\therefore$ Eq (1) can be rearranged as:

$$\frac{B}{A}\left[\frac{(1 - i\alpha_2)}{Z_2} + \frac{(1 + i\alpha_1)}{Z_1}\right] = \frac{(1 - i\alpha_1)}{Z_1} - \frac{(1 - i\alpha_2)}{Z_2} \qquad -(2)$$

$$\alpha_2 = \frac{1}{\frac{\omega}{c_2} a} = \frac{1}{\left(\frac{\omega}{c_1} a\right)\frac{c_1}{c_2}} = \frac{\alpha}{\frac{Z_1}{Z_2}\left(\frac{\rho_2}{\rho_1}\right)}$$

Substituting this value of α_2 in Eq (2) we get

$$\frac{B}{A}\left[\frac{(1 - i\alpha_2)}{Z_2} + \frac{(1 + i\alpha_1)}{Z}\right] = \frac{1 - i\alpha_1}{Z} - \frac{\left(1 - i\alpha_1 \frac{Z_2}{Z}\frac{\rho_1}{\rho_2}\right)}{Z_2}$$

$$\therefore \frac{B}{A} = \frac{\frac{1}{\cancel{Z_1}}\left\{\left(1-\frac{Z_1}{Z_2}\right) - i\alpha_1\left(1-\frac{\rho_1}{\rho_2}\right)\right\}}{\frac{1}{\cancel{Z_1}}\left\{1+\frac{Z_1}{Z_2} + i\alpha_1\left(1-\frac{\rho_1}{\rho_2}\right)\right\}}$$

[60%]

c) For a compact balloon

$k_1 a \ll 1 \Rightarrow \alpha_1 \gg 1$.

$$\therefore \frac{B}{A} \sim -1$$

$$\Rightarrow \alpha \sim 0 \quad \text{or} \quad \boxed{p_t' \sim 0}$$

[20%]

Examiner's note:

(a) This was answered very well by most students. The average mark for this question is very high. Some candidates lost marks for not explaining their reasoning.

(b) A few students got stuck in the second part (about 15%) as they were not sure of the boundary conditions to apply on the surface of the balloon.

2 (a) From the datacard

$$\Box^2 \rho' = \frac{\partial^2 T_{ij}}{\partial x_i \partial x_j}$$

Using the Green's f^n for the wave-equation from the data card, the sound field from a single eddy is given by

$$\rho' = \iint \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \frac{\delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\}}{4\pi c_0 |\underline{x}-\underline{y}|} \, d\underline{y} \, d\tau$$

In the far field,

$$\frac{1}{|\underline{x}-\underline{y}|} \sim \frac{1}{|\underline{x}|} = \frac{1}{x}$$

$$\rho' \sim \frac{1}{4\pi c_0 x} \iint \frac{\partial^2 T_{ij}}{\partial y_i \partial y_j} \delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\} \, d\underline{y} \, d\tau$$

Using a property of convolution integrals,

$$\rho' = \frac{1}{4\pi c_0 x} \frac{\partial^2}{\partial x_i \partial x_j} \iint T_{ij} \, \delta\{|\underline{x}-\underline{y}| - c_0(t-\tau)\} \, d\underline{y} \, d\tau.$$

Using the property of $\delta()$ function to integrate with respect to τ,

$$\rho = \frac{1}{4\pi c_0^2 x} \frac{\partial^2}{\partial x_i \partial x_j} \iint T_{ij}\left(\underline{y}, t - \frac{|\underline{x}-\underline{y}|}{c_0}\right) d\underline{y}$$

⑥

For a compact eddy, we can neglect retarded time differences. Therefore,

$$\rho' = \frac{1}{4\pi c_0^2 x} \frac{\partial^2}{\partial x_i \partial x_j} \iiint T_{ij}\left(\underline{y}, t - \frac{x}{c_0}\right) d\underline{y}$$

Using the hint:

$$\rho' = \frac{1}{4\pi c_0^2 x} \frac{1}{(c_0)^2} \left(\frac{x_i x_j}{x^2}\right) \frac{\partial^2}{\partial t^2} \int T_{ij}\left(\underline{y}, t - \frac{x}{c_0}\right) d\underline{y}$$

with $T_{ij} = \rho_0 u_i u_j$

Let the length scale of the eddy be l, velocity scale u', then

$$\rho' \sim \frac{1}{c_0^4 x} \beta_i \beta_j \frac{u'^2}{l^2} \rho_0 u_0'^2 l^3$$

$$= \rho_0 \underline{\beta_i \beta_j} \left(\frac{l}{x}\right) M^4$$

[60%]

2(b) The power radiated from a single eddy is

$$P \sim p' u' x^2 = \frac{p'^2}{\rho_0 c_0} x^2 = \frac{c_0^4 \rho'^2 x^2}{\rho_0 c_0}$$

$$\therefore P \sim \frac{c_0^3 \cancel{x^2}}{\cancel{\rho_0}} \cdot \rho_0^{\cancel{2}} (\beta_i \beta_j)^2 \frac{l^2}{\cancel{x^2}} M^8$$

$$= \rho_0 (\beta_i \beta_j)^2 \frac{l^2}{c_0^5} u'^5$$

Assume that the eddies are randomly distributed, therefore we loose directionality.
& u' scales as u_j and $l \sim d_j$

$$\therefore \boxed{P_{jet} = \frac{\rho_0 d_j^2}{c_0^5} u_j^8}$$

[40%]

Examiner's note:

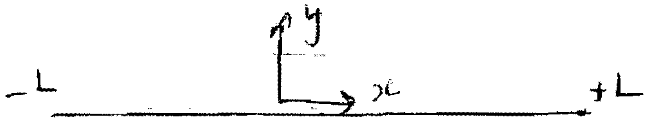
(a) This was answered well by most students

(b) A common mistake made by about a third of the students was to start with an incorrect definition of a convolution integral. This did not materially affect their solution but they lost a couple of marks.

(c) A few students lost marks for stating that the retarded time differences can be dropped because the observer is in the far field. Again this did not affect their final solution as retarded time differences need to be dropped albeit for a different reason (compact sources).

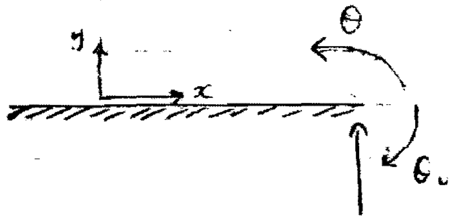
(d) About a third of the students lost marks in the final part because either they go the definition of power wrong or they did not explain that the length and velocity scales of an eddy scales respectively with the jet diameter and exit velocity.

~~(b)~~

3. 

(a) $\uparrow p_i$

Consider edge $x = +L$ first



$$\theta_i = -\pi/2$$

$$\theta = \pi - \tan^{-1}(y/L)$$

$$r = \sqrt{L^2 + y^2}$$

So using formula from data card.
diffracted field $\rightarrow$

$$p_i \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(L^2+y^2)^{1/4}} \frac{\sin\left(-\frac{\pi}{4}\right) \sin\left(\frac{\pi}{2} - \frac{1}{2}\tan^{-1}(y/L)\right)}{\cos\left[\pi - \tan^{-1} y/L\right] + \cos\left[-\frac{\pi}{2}\right]} e^{-\frac{i\pi}{4} - ik_0 r}$$

$$= p_i \left(\frac{2}{\pi}\right)^{1/2} \frac{1}{(L^2+y^2)^{1/4}} \frac{-\frac{1}{2^{1/2}} \cos\left[\frac{1}{2}\tan^{-1}(y/L)\right]}{-1 \cos\left(\tan^{-1} y/L\right)} e^{-\frac{i\pi}{4} - ik_0 r}$$

$$= p_i \left(\frac{1}{\pi}\right)^{1/2} \frac{1}{(L^2+y^2)^{1/4}} \frac{\cos\left[\frac{1}{2}\tan^{-1} y/L\right]}{\frac{L}{(L^2+y^2)^{1/2}}} e^{-i\pi/4 - ik_0\sqrt{L^2+y^2}}$$

$$= \frac{p_i (L^2+y^2)^{1/4}}{\pi^{1/2} L} \cos\left[\frac{1}{2}\tan^{-1} y/L\right] e^{-i\pi/4 - ik_0\sqrt{L^2+y^2}}$$

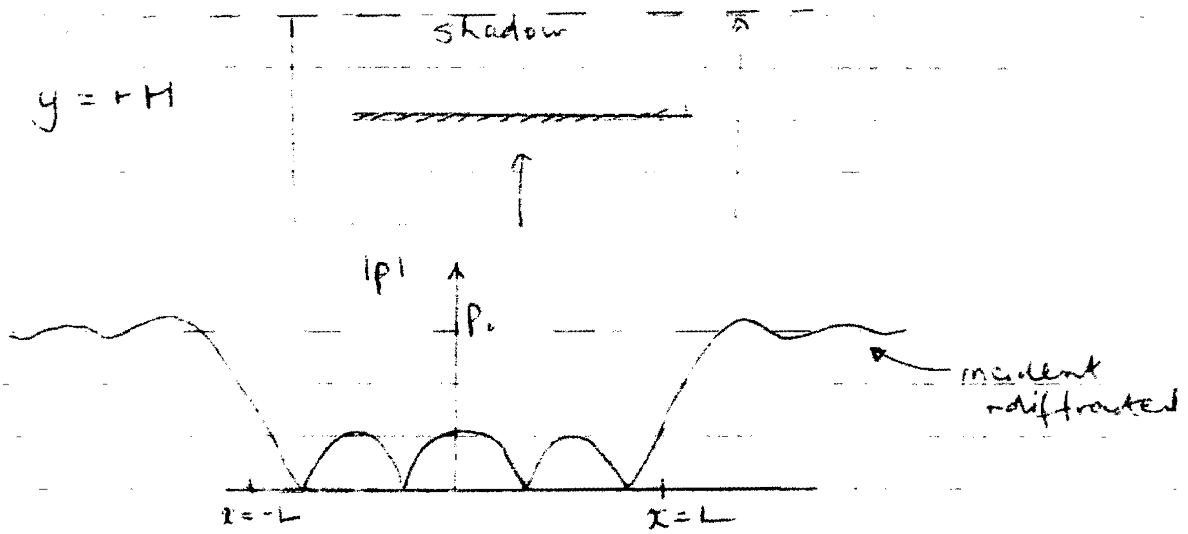
By symmetry, contribution from $x = -L$ is same, so

total pressure is

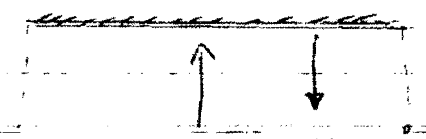
$$\frac{2p_i\,(L^2+y^2)^{1/4}}{\pi^{1/2}\,L}\cos\left[\frac{1}{2}\tan^{-1}\left(\frac{y}{L}\right)\right]e^{-\frac{i\pi}{4}-ik_0\sqrt{L^2+y^2}}$$

[40%

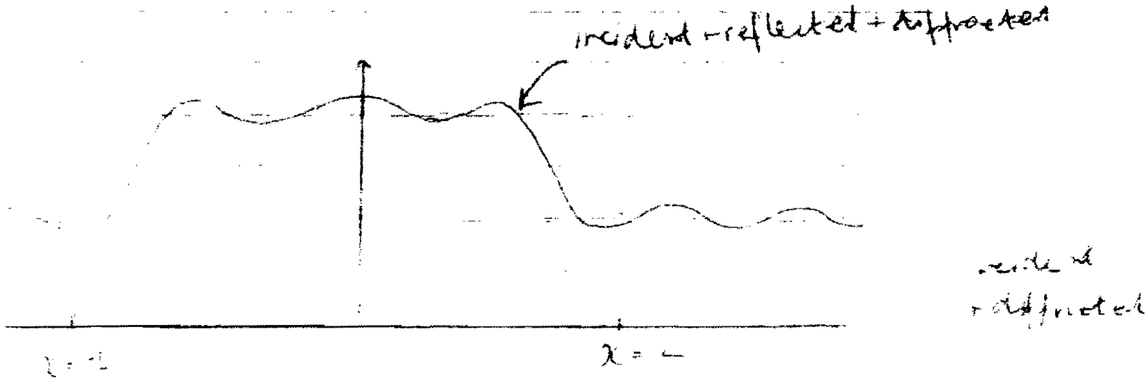
(b) $y = +H$



$y = -H$

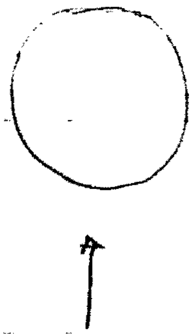


reflected field $= p_i\, e^{i\omega t + ik_0 y}$



40%

(c)



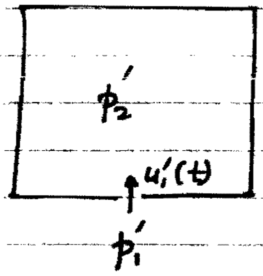
- shadow is much deeper than for sharp edges, so small amplitude in $|x| < L$, $y = +H$
- additional contribution along $y = -H$ due to creeping rays

[20%]

Examiner's note:

(a) Most attempts at this part of the question were quite reasonable, although a number of candidates failed to interpret the correct value of the incident angle. Few candidates realised that the scattered pressure from a single edge could be doubled along the line of symmetry.
(b) The main error here was in failing to remember to add in the reflected pressure below the plate.
(c) Generally fair answers to this part.

4 a)



$$p_1' - p_2' = \rho_0\left(l\frac{\partial u_1'}{\partial t} + c_0\alpha u_1'\right) = \rho_0\left(li\omega + c_0\alpha\right)u_1'$$

for disturbances of frequency ω.

$\rho_0 A u_1'$ = rate of change of mass of air in the bulb

$$= V\frac{\partial \rho_2'}{\partial t} = \frac{V}{c_0^2}\frac{\partial p_2'}{\partial t} = \frac{V\,i\omega}{c_0^2}\,p_2'(t)$$

Hence $\quad p_1'(t) = p_2'(t) + \rho_0(li\omega + c_0\alpha)u_1'$

$$p_1'(t) = \rho_0\left(\frac{c_0^2 A}{Vi\omega} + li\omega + c_0\alpha\right)u_1'(t) \qquad [40\%]$$

b) $\quad p_1'(t) = (I+R)e^{i\omega t}$

mass flow / unit wall area $= \frac{1}{c_0}(I-R)e^{i\omega t}$

$= \rho_0\,A\,u_1'(t) \times$ number of holes / unit area

$= \rho_0 u_1'(t)\,\beta$

Hence applying (1) $\quad I+R = \frac{1}{\beta}\left(\frac{c_0^2 A}{Vi\omega} + li\omega + c_0\alpha\right)\frac{(I-R)}{c_0}$

For total absorption $\quad R = 0$,

this requires $\quad \frac{1}{\beta c_0}\left(\frac{c_0^2 A}{Vi\omega} + li\omega + c_0\alpha\right) = 1$

Equating real parts $\quad \frac{c_0\alpha}{\beta c_0} = 1 \quad$ i.e. $\quad \beta = \alpha$

porosity $= \underline{\underline{\alpha = 0.1}}$

Equating imaginary parts $\quad \frac{c_0^2 A}{Vi\omega} + li\omega = 0$

i.e. $\quad \frac{c_0^2 A}{lV} = \omega^2$

$A = \pi a^2$, $\quad l = 1.2\,a$, $\quad V = \frac{d}{N} = Ad/\beta \quad$ where N = number of Helmholtz resonators / unit area

Hence $\frac{c_0^2 A \beta}{1.2\, a\, d\, A} = \omega^2$

rearranging $a = \frac{c_0^2}{\omega^2} \frac{\beta}{1.2\, d} = \left(\frac{343}{2\pi 750}\right)^2 \frac{0.1}{1.2 \times 5 \times 10^{-2}}$

$\underline{\underline{a = 8.8 \times 10^{-3}\ m}}$ [60%]

i.e hole diameter 0.0177m.

Examiner's note

(a) Most students were able to do this part correctly.
(b) A common mistake was to try to maximise the energy absorbed by the liner, which leads to complicated algebra, rather than using the simpler approach given in the question of setting the reflected wave to zero. Very few students got this question completely correct.