1. (a) Differences:

(i) Obvious time delay in both signals, due to the filters

(ii) The force signal had a rounded top but in the digitised version it is flat — it has presumably been clipped because the peak amplitude was too big for the filters

(iii) The output signal is no longer zero before the hammer pulse. A low-amplitude oscillation is visible, revealed to be 50 Hz mains hum by the transfer function

(iv) The hammer pulse does not stop when contact is lost. A decaying oscillation is visible. This will be Gibbs' phenomenon from the filter: the "true" force signal has sharp corners in it, and hence has significant signal at high frequencies. These will be removed by the filter, leaving a characteristic ripple at around the cut-off frequency of the filter.

(v) Note that both scales are in digital units, and the numbers are rather small, so digitisation errors are to be expected. In particular, the low-amplitude ripple in the force signal by the end of the range is giving just one digital unit — i.e. it has been turned into a square wave.

40%

(b) Mains-hum peak at 50 Hz is obvious. Signal after 75 Hz or so is very noisy, because the filters have brought both input and output to low levels. Thus one small number is being divided by another to calculate the transfer function, leading to a noisy result.

30%

1 cont.

(c) Measuring from the transfer function, peak occurs at 25.5 mm, where 100 Hz is 81 mm.

So frequency is $100 \times \frac{25.5}{81} \simeq 31.5$ Hz

So $\omega = 2\pi f = \sqrt{\frac{k}{m}}$, and $m = 1$ kg

So $k = (2\pi \times 31.5)^2 \simeq 3.9 \times 10^4$ N/m

Best to get damping from logarithmic decrement.

Decay in one cycle is 12 mm → 9 mm measured from 1(c).

So $\zeta \simeq \frac{1}{2\pi} \ln \frac{y_1}{y_2} \simeq 0.05$

30%

Popular and well done in general. Few noticed the values on the scale of "digital units" showing that both signals have very poor dynamic range. Most tried to estimate damping via 3 dB bandwidth, hard to see accurately on this scale, whereas the log dec estimated from the time history gives better accuracy.

2(a) Try $w = X(x)\,Y(y)\,e^{i\omega t}$. Then equation requires

$$D\left(X''''Y + 2X''Y'' + XY''''\right) - m\omega^2 XY = 0$$

Need to manipulate so each term contains x or y but not both. Must divide through by XY for 1st and 3rd terms to work:

$$\frac{X''''}{X} + \frac{2X''}{X}\frac{Y''}{Y} + \frac{Y''''}{Y} - \frac{m\omega^2}{D} = 0 \qquad (1)$$

In order for middle term to work, one or other of $\frac{X''}{X}$, $\frac{Y''}{Y}$ must be constant, as required.

25%

(b) If $\frac{X''}{X} = -k_1^2$ so that $X = A\sin k_1 x + B\cos k_1 x$,

then $\frac{X''''}{X} = k_1^4$.

Now eq. (1) gives $\frac{X''''}{X} - \frac{m\omega^2}{D} = -\frac{Y''''}{Y} + 2k_1^2\frac{Y''}{Y} = \text{const}$

and from the first part, the constant $= k_1^4 - \frac{m\omega^2}{D}$

So the equation for Y is

$$Y'''' - 2k_1^2 Y'' + \left(k_1^4 - \frac{m\omega^2}{D}\right)Y = 0$$

25%

(c) Try $X = \sin\left(\frac{p\pi x}{L_1}\right)$, $Y = \sin\left(\frac{q\pi y}{L_2}\right)$

Then $\frac{X''}{X} = -\left(\frac{p\pi}{L_1}\right)^2 = -k_1^2$

So Y equation from (b) gives

$$\left(\frac{q\pi}{L_2}\right)^4 + 2\left(\frac{p\pi}{L_1}\right)^2\left(\frac{q\pi}{L_2}\right)^2 + \left(\frac{p\pi}{L_1}\right)^4 = \frac{m\omega^3}{D}$$

2(c) cont: so natural frequencies are given by

$$\omega^2 = \frac{D}{m}\left[\left(\frac{p\pi}{L_1}\right)^2 + \left(\frac{q\pi}{L_2}\right)\right]^2$$

Boundary conditions: the sinusoidal shapes automatically satisfy conditions at $x = 0$, $y = 0$

At $x = L_1$, need $\sin\left(\frac{p\pi x}{L_1}\right) = 0$

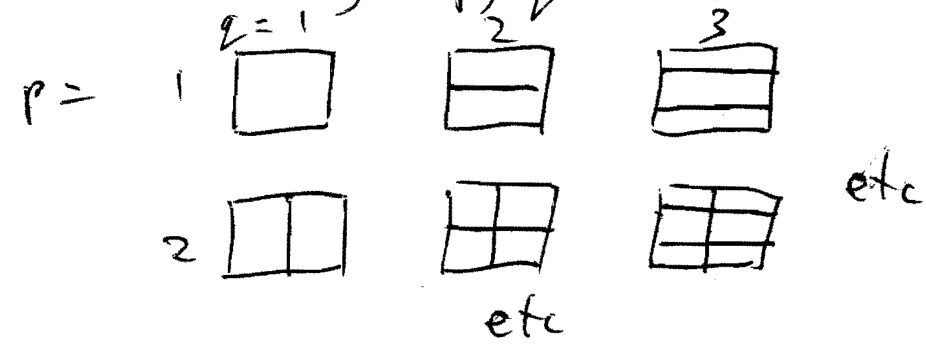
so $p = 1, 2, 3 \ldots$. $X''(L_1) = 0$ automatically.

Similarly, at $y = L_2$ need $\sin\left(\frac{q\pi y}{L_2}\right) = 0$

25/2

so $q = 1, 2, 3 \ldots$

(d) Numbers of nodal lines in x, y directions are determined by p, q values:



Mass at the centre will reduce the frequency of all modes that don't have a nodal point there. (Rayleigh's principle says adding mass must reduce frequencies) So modes with p, q both odd numbers are shifted, all others stay the same.

By the interlacing theorem, a mode can't be shifted further than the next mode below that frequency. High mass will pin the centre point.

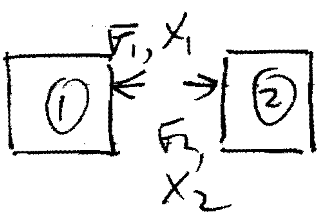
25/2

Examiner's comment:

The most popular question. The most common failing was in part (b): the given constant k_1 defines the value of the separation constant, and the equation for Y should not contain a second undefined constant. Many failed to notice that the question does not say that p and q in part (c) are integers: that is to be deduced from the boundary conditions.

5

3(a)



$$Y_1 = \frac{X_1}{F_1}, \quad Y_2 = \frac{X_2}{F_2}$$

Couple: $X_1 = -X_2 = X$ say, total $F (\rightarrow) = F_2 - F_1$

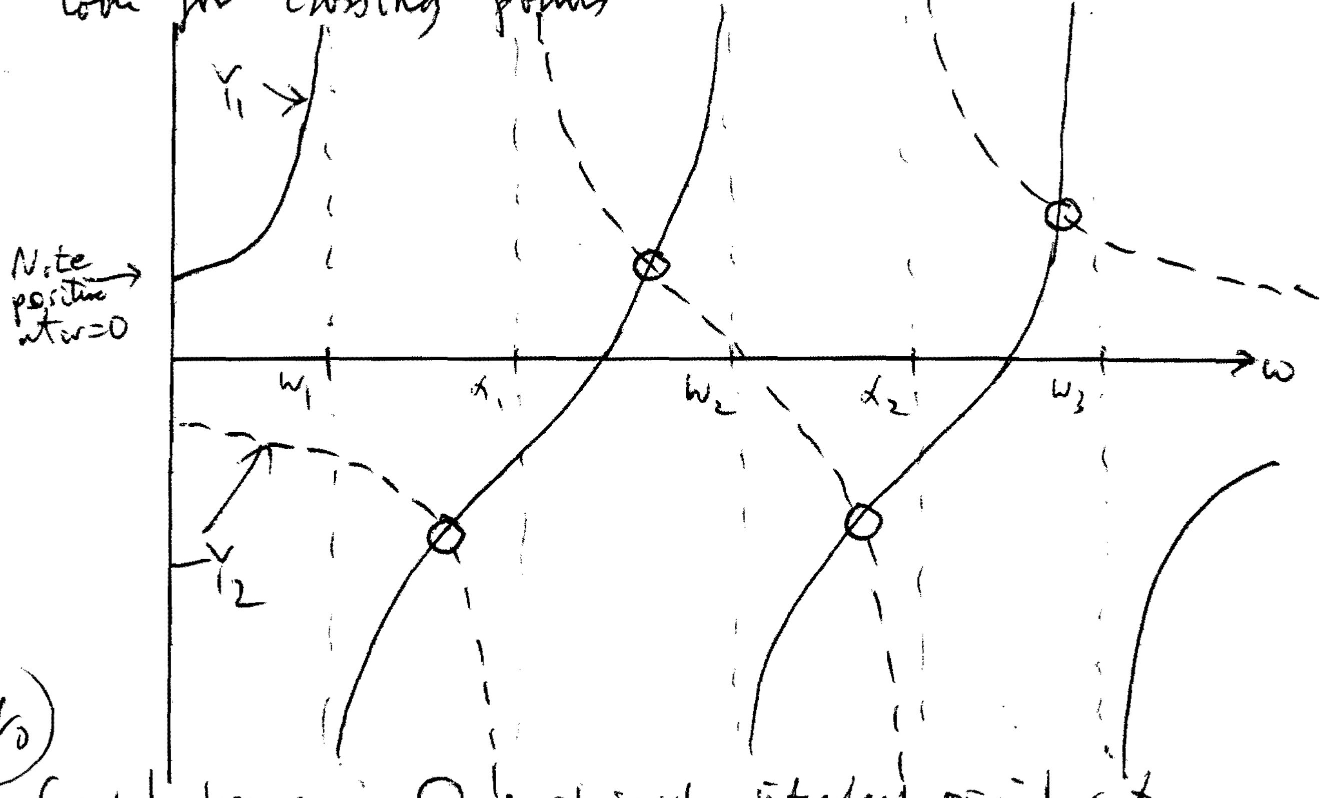
so $F = F_2 - F_1 = \frac{X_2}{Y_2} - \frac{X_1}{Y_1} = X\left(\frac{1}{Y_1} + \frac{1}{Y_2}\right)$

But $\frac{X}{F} = Y_{coupled}$ by definition, so $\frac{1}{Y_{coupled}} = \frac{1}{Y_1} + \frac{1}{Y_2}$

At a natural frequency $Y \to \infty$ so $\frac{1}{Y} \to 0$

15% So coupled natural frequencies where $\frac{1}{Y_1} + \frac{1}{Y_2} = 0$ //

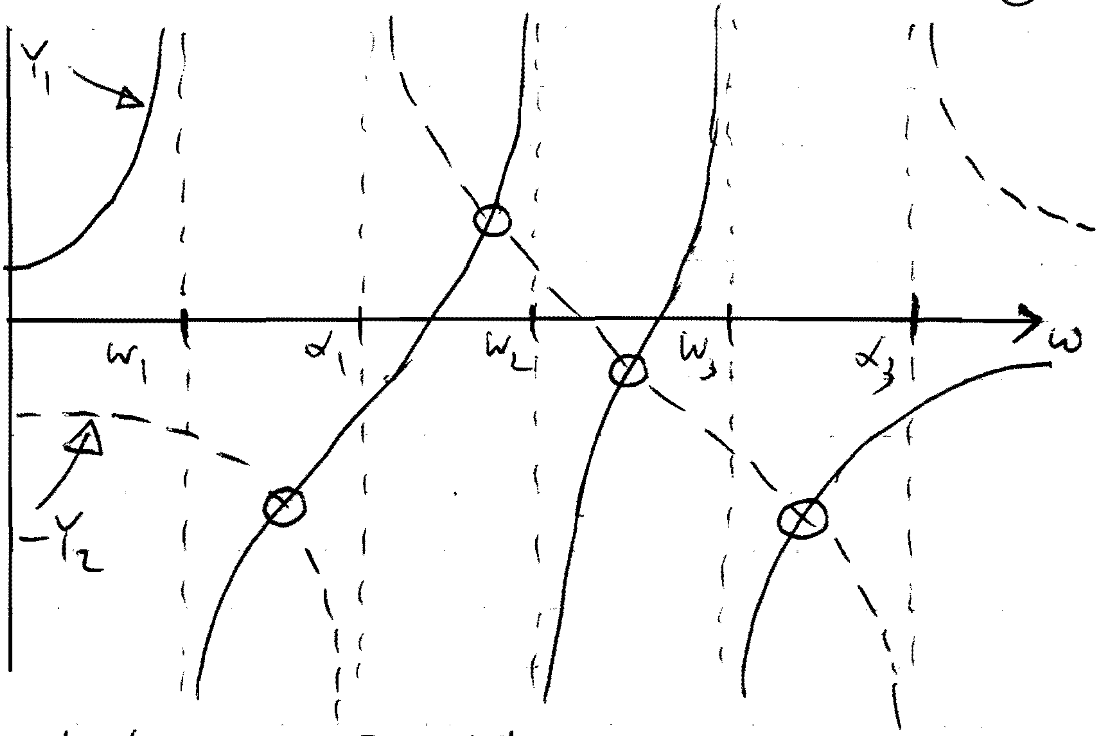
(b) No damping, so Y_1, Y_2 both real. Plot on a linear scale: signs matter so dB scale no good. From (a) need $Y_1 = -Y_2$, so plot Y_1 & $-Y_2$ and look for crossing points.



30%

Coupled frequencies O : obviously interlace original set.

3 (c)



25% 4 coupled frequencies O still interlace.

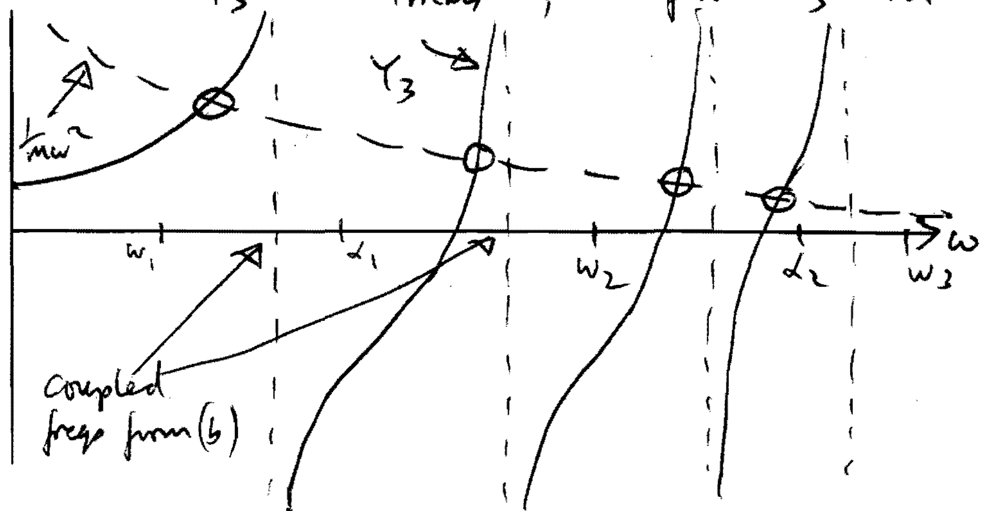
(d) Define coupled system from (b) as Y_3:

$$\frac{1}{Y_3} = \frac{1}{Y_1} + \frac{1}{Y_2}$$

Now add mass, to get $\dfrac{1}{Y_{coup}} = \dfrac{1}{Y_3} + \dfrac{1}{Y_{mass}}$

For mass $m\ddot{x} = f$, so $F = -m\omega^2 X$, $\therefore Y_{mass} = X/F = -1/m\omega^2$

Now want $Y_3 = -Y_{mass}$, so plot Y_3 and $1/m\omega^2$:



30% Frequencies interlace from (b), all reduced, but don't necessarily interlace original set ω_i, α_i. "Mass" has frequency zero on its own.

4(a) Kinetic energy $T = \frac{1}{2} m \int \dot{w}^2 \, dx$

Potential energy

$$V = \frac{1}{2} EI \int_0^L w''^2 \, dx + \frac{1}{2}\lambda \left[w'(0)\right]^2 + \frac{1}{2}\lambda\left[w'(L)\right]^2$$

Now $E \to E(1+i\eta)$, $\lambda \to \lambda(1+i\gamma)$, $w \propto \sin\frac{n\pi x}{L}$

Rayleigh quotient:

$$\omega^2 \simeq \frac{\frac{1}{2} EI(1+i\eta)\int_0^L \left(\frac{n\pi}{L}\right)^4 \sin^2\frac{n\pi x}{L}\, dx + \frac{1}{2}\lambda(1+i\gamma)\left[\left(\frac{n\pi}{L}\right)^2 + \left(\frac{n\pi}{L}\right)^2\right]}{\frac{1}{2} m \int_0^L \sin^2 \frac{n\pi x}{L} \, dx}$$

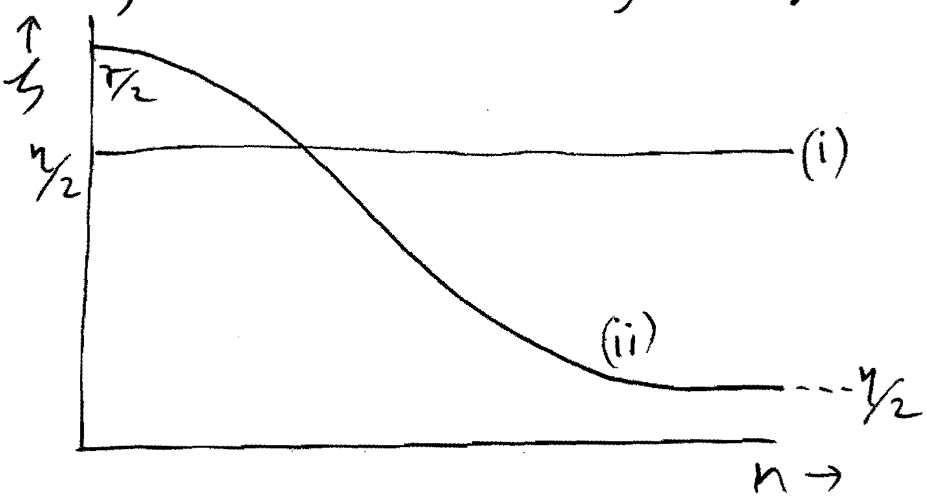
$$= \frac{\frac{1}{2} EI(1+i\eta)\left(\frac{n\pi}{L}\right)^4 \cdot L/2 + \lambda(1+i\gamma)(n\pi/L)^2}{\frac{1}{2} m \cdot L/2}$$

$$= \frac{EI}{m}\left(\frac{n\pi}{L}\right)^4 (1+i\eta) + \frac{4\lambda}{mL}\left(\frac{n\pi}{L}\right)^2 (1+i\gamma)$$

Damping ratio $\zeta = \frac{1}{2Q} = \frac{1}{2}\frac{\mathrm{Im}(\omega^2)}{\mathrm{Re}(\omega^2)} = \frac{EI\left(\frac{n\pi}{L}\right)^2 \eta + \frac{4\lambda}{L}\gamma}{2\left[EI\left(\frac{n\pi}{L}\right)^2 + \frac{4\lambda}{L}\right]}$

(i) If $\eta = \gamma$, $\zeta = \eta/2$ independent of n

(ii) If $\eta \ll \gamma$, γ term wins at low n but η term eventually takes over as n gets large



35%

(b) Now $V = \frac{1}{2} EI(1+i\eta)\int_0^L w''^2 dx + \frac{1}{2} P \int_0^L w'^2 dx$

Note no damping associated with P term.
Use $w = \sin\frac{n\pi x}{L}$ again, & do same calculation

$$\omega^2 \simeq \frac{\frac{1}{2}EI\left(\frac{n\pi}{L}\right)^4(1+i\eta)\cdot\frac{L}{2} + \frac{1}{2}P\left(\frac{n\pi}{L}\right)^2\cdot L/2}{\frac{1}{2}m\cdot L/2}$$

$$= \left(\frac{n\pi}{L}\right)^2\left[\frac{P}{m} + \frac{EI}{m}\left(\frac{n\pi}{L}\right)^2(1+i\eta)\right]$$

So $\zeta \simeq \frac{\mathrm{Im}(\omega^2)}{2\mathrm{Re}(\omega^2)} \simeq \frac{EI\left(\frac{n\pi}{L}\right)^2\eta}{2\left[P + EI\left(\frac{n\pi}{L}\right)^2\right]}$: so P makes ζ lower

30%

(c) Pressure in fuselage is analogous effect to (b) above: it increases potential energy but does not add to Im(V), and so ζ tends to reduce, as observed.

There are other physical effects too: pressure may increase the locking force on riveted or bolted joints, and hence reduce the amount of micro-slip: again, stiffer and lower ζ
Damping in a fuselage comes from material effects, especially the added interior trim. Also from boundary effects at joints, bolted seat/floor connections and other internal fittings like luggage compartments.

Damping is already added by trim, and this can be enhanced by design details: material choose, fixings etc. Hard to add much more damping without weight penalties, but some aircraft use active dampers attached to skin panels: feedback loop senses motion and feeds back force simulating attached dashpot.

35%

Examiner's comment:

The majority did part (a) quite competently. For part (b) many took the equation for axial vibration of a rod, rather for lateral vibration of a tensioned string, as the way to represent the effect of tension. Part (c) produced some good general discussions of damping mechanisms in an aircraft structure, but almost no-one noticed the link between parts (b) and (c) in giving one mechanism for damping to fall with pressurization. Rather few seem to have noticed that aircraft have interior trim covering the metal panels, which has a big effect on damping!

4C6 2012 Answers

1(c) Stiffness around 40 kN/m; loss factor around 0.05.

2(b) $Y'''' - 2k_1^2 Y'' + \left(k_1^4 - \frac{m\omega^2}{D}\right)Y = 0$

(c) $\omega = \sqrt{\frac{D}{m}}\left[\left(\frac{p\pi}{L_1}\right)^2 + \left(\frac{q\pi}{L_2}\right)^2\right]$

4(a) $\omega^2 \approx \frac{EI}{m}\left(\frac{n\pi}{L}\right)^4 + \frac{4\lambda}{mL}\left(\frac{n\pi}{L}\right)^2$; damping ratio $\approx \dfrac{EI\left(\frac{n\pi}{L}\right)^2 \eta + \frac{4\lambda}{L}\gamma}{2\left[EI\left(\frac{n\pi}{L}\right)^2 + \frac{4\lambda}{L}\right]}$

(b) $\omega^2 \approx \frac{EI}{m}\left(\frac{n\pi}{L}\right)^4 + \frac{P}{m}\left(\frac{n\pi}{L}\right)^2$; damping ratio $\approx \dfrac{EI\left(\frac{n\pi}{L}\right)^2 \eta}{2\left[EI\left(\frac{n\pi}{L}\right)^2 + P\right]}$