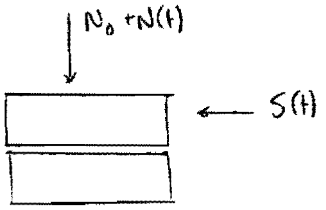
1) a)



For slip $|S(t)| \geq \mu(N_0 + N)$

For $S(t) > 0 \Rightarrow S(t) \geq \mu(N_0 + N)$

$\Rightarrow \underline{S(t) - \mu N(t) \geq \mu N_0}$

For $S(t) < 0 \Rightarrow -S(t) \geq \mu(N_0 + N)$

$\Rightarrow \underline{S(t) + \mu N(t) \leq -\mu N_0}$ [15%]

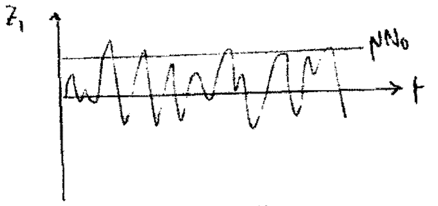
b) Put $z_1 = S(t) - \mu N(t) \Rightarrow z_1^2 = S^2 + \mu^2 N^2 - 2\mu N S$

$E[z_1^2] = \sigma_a^2 + \mu^2 \alpha \sigma_a^2 + 0$, noting statistical independence

$\underline{\sigma_{z_1}^2 = \sigma_a^2 (1 + \mu^2 \alpha)}$

Also $\dot{z}_1 = \dot{S} - \mu \dot{N} \Rightarrow \underline{\sigma_{\dot{z}_1}^2 = \sigma_b^2 (1 + \mu^2 \beta)}$

Slip when $z_1 \geq \mu N_0$



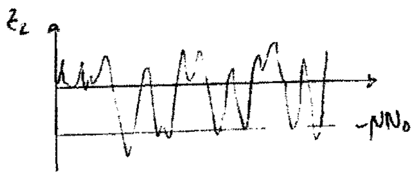
Rate of slip due to z_1 = rate of crossing μN_0

$$\nu_1 = \frac{1}{2\pi} \left(\frac{\sigma_{\dot{z}_1}}{\sigma_{z_1}} \right) e^{-\frac{1}{2} \left(\frac{\mu N_0}{\sigma_{z_1}} \right)^2}$$

Also $z_2 = S(t) + \mu N(t) \Rightarrow \sigma_{z_2}^2 = \sigma_a^2 (1 + \mu^2 \alpha)$

$\sigma_{\dot{z}_2}^2 = \sigma_b^2 (1 + \mu^2 \beta)$

Slip when $z_2 \leq -\mu N_0$



Rate of slip due to z_2 = rate of crossing $-\mu N_0$

$$\nu_2 = \frac{1}{2\pi} \left(\frac{\sigma_{\dot{z}_2}}{\sigma_{z_2}} \right) e^{-\frac{1}{2} \left(\frac{\mu N_0}{\sigma_{z_2}} \right)^2}$$

b) cont

Combined rate of slip: $\nu_1 + \nu_2 = \underline{\left(\frac{1}{\pi}\right)\left(\frac{1+\mu^2\beta}{1+\mu^2\alpha}\right)^{\frac{1}{2}}\left(\frac{\sigma_b}{\sigma_a}\right) e^{-\frac{1}{2}\left[\frac{(\mu N_0)^2}{(1+\mu^2\alpha)\sigma_a^2}\right]}}$ [50%]

c) For $N(t) = \gamma S(t)$, $z_1 = S - \mu N = S(1-\mu\gamma)$

$$\Rightarrow \quad \underline{\sigma_{z_1}^2 = (1-\mu\gamma)^2 \sigma_a^2}$$

$$\underline{\sigma_{\dot{z}_1}^2 = (1-\mu\gamma)^2 \sigma_b^2}$$

$$\Rightarrow \quad \underline{\nu_1 = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_b}{\sigma_a}\right) e^{-\frac{1}{2}\left[\frac{(\mu N_0)^2}{(1-\mu\gamma)^2\sigma_a^2}\right]}}$$

Same analysis for z_2, but with $z_2 = S + \mu N = S(1+\mu\gamma)$ $\Rightarrow$ $\nu_2 = \nu_1$ with $(1-\mu\gamma)$ replaced by $(1+\mu\gamma)$

$$\Rightarrow \quad \underline{\nu_1 + \nu_2 = \left(\frac{1}{2\pi}\right)\left(\frac{\sigma_b}{\sigma_a}\right)\left\{ e^{-\frac{1}{2}\left[\frac{(\mu N_0)^2}{(1-\mu\gamma)^2\sigma_a^2}\right]} + e^{-\frac{1}{2}\left[\frac{(\mu N_0)^2}{(1+\mu\gamma)^2\sigma_a^2}\right]} \right\}}$$ [15%]

Examiner's comment:

This question was the least popular on the paper, no doubt because the topic (slipping under friction) has not appeared on any previous Tripos paper. However, those students who attempted the question generally produced very impressive answers, and the question obtained the highest average mark for the paper.

2) a) $\ddot{x} + 2\beta_n \omega_n \dot{x} + \omega_n^2 x = F(t)$

Put $x(t) = x(\omega)e^{i\omega t}$ and $F(t) = F(\omega)e^{i\omega t}$

$$(-\omega^2 + 2i\beta_n\omega_n\omega + \omega_n^2)\,x(\omega) = F(\omega)$$

$$\Rightarrow \quad x(\omega) = \underbrace{\left[\frac{1}{\omega_n^2-\omega^2+2i\beta_n\omega_n\omega}\right]}_{\text{Frequency response function } H(\omega)} F(\omega)$$

$$S_{xx}(\omega) = |H(\omega)|^2 S_{FF}(\omega)$$

$$\underline{S_{xx}(\omega) = \frac{S_0}{(\omega_n^2-\omega^2)^2+(2\beta_n\omega_n\omega)^2}}$$ [15%]

b) $\ddot{p} + 2\beta_a\omega_a\dot{p} + \omega_a^2 p = \alpha\ddot{x}(t)$

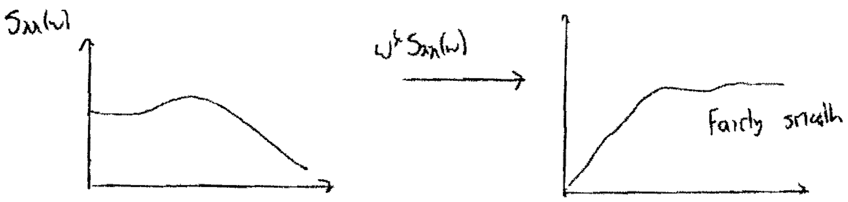
Put $p(t) = p(\omega)e^{i\omega t}$ and $x(t) = x(\omega)e^{i\omega t}$

$$\Rightarrow \quad p(\omega) = \underbrace{\left[\frac{-\alpha\omega^2}{(\omega_a^2-\omega^2)+2i\beta_a\omega_a\omega}\right]}_{\text{Frequency response function } H_p(\omega)} x(\omega)$$

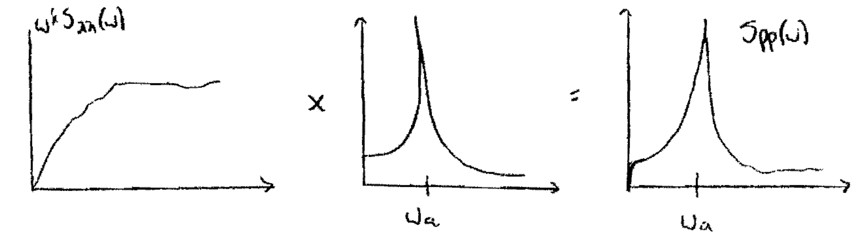
$$S_{pp}(\omega) = |H_p(\omega)|^2 S_{xx}(\omega)$$

$$\Rightarrow \quad \underline{S_{pp}(\omega) = \alpha^2\omega^4 S_0\left[\frac{1}{(\omega_n^2-\omega^2)^2+(2\beta_n\omega_n\omega)^2}\right]\left[\frac{1}{(\omega_a^2-\omega^2)^2+(2\beta_a\omega_a\omega)^2}\right]}$$ [20%]

c) β_n large, β_a small $\longrightarrow$



$S_{pp}(\omega) \longrightarrow$



White noise approximation $\longrightarrow$ approximate $\omega^4 S_{xx}(\omega)$ to $\omega_a^4 S_{xx}(\omega_a)$

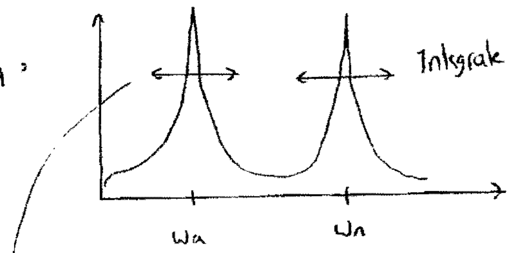
Acoustic response to white noise $\quad \sigma_p^2 = \dfrac{\pi}{4\beta_a \omega_a^3} \times \underbrace{\text{white noise spectrum}}_{\alpha^2 S_0 \omega_a^4 S_{xx}(\omega_a)}$

$$\sigma_p^2 = \frac{\pi \alpha^2 S_0 \omega_a}{4\beta_a} \left[(\omega_n^2 - \omega_a^2)^2 + (2\beta_n \omega_n \omega_a)^2 \right]^{-1}$$ [25%]

d) Similar to part (c) but now $\omega^4 |H_p(\omega)|^2$ approximated to $\omega_n^4 |H_p(\omega_n)|^2$, and response becomes

$$\sigma_p^2 = \frac{\pi \alpha^2 S_0 \omega_n}{4\beta_n} \left[(\omega_n^2 - \omega_a^2)^2 + (2\beta_a \omega_a \omega_n)^2 \right]^{-1}$$ [20%]

e) $S_{pp}(\omega)$:



Integrate here, assuming H_p smooth over this region

Integrate here assuming H smooth over this region.

Result is the sum of (d) and (c), noting that $(\omega_n^2 - \omega_a^2)^2 + (2\beta_a \omega_a \omega_n)^2 \approx (\omega_n^2 - \omega_a^2)^2$.

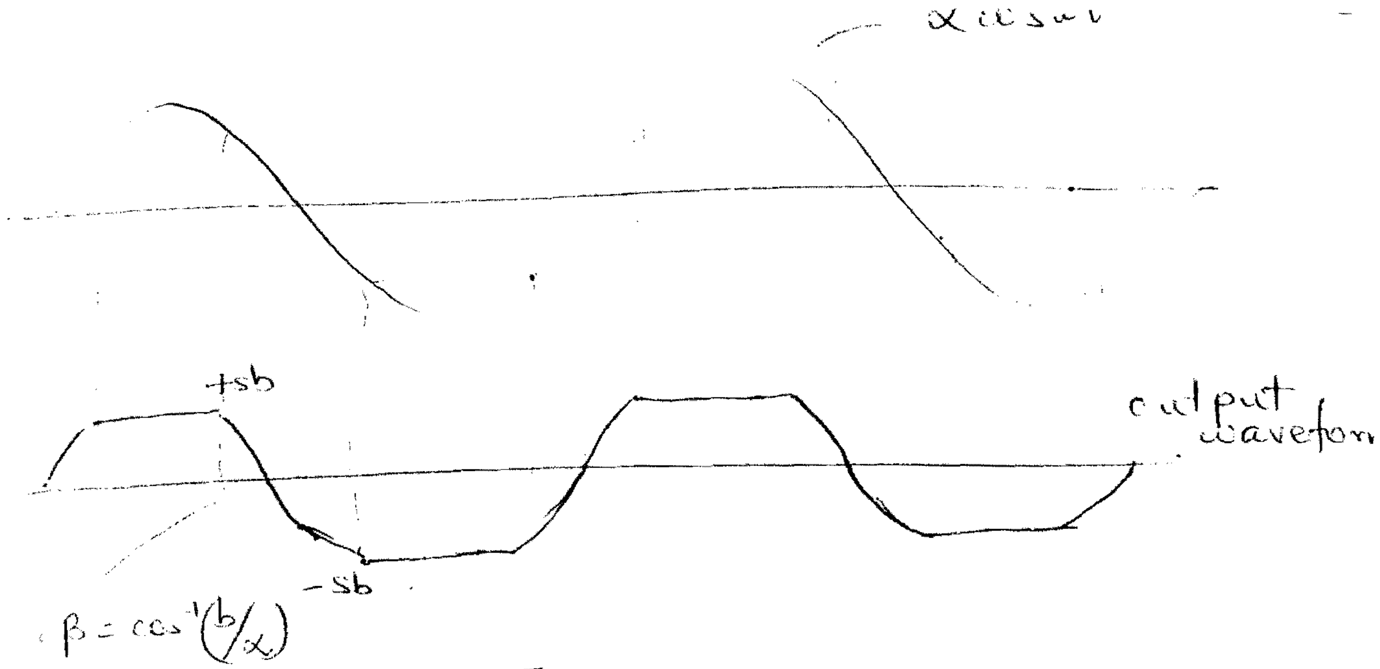
Thus $$\sigma_p^2 \approx \frac{\pi \alpha^2 S_0}{4} \left(\frac{\omega_a}{\beta_a} + \frac{\omega_n}{\beta_n} \right) \left(\frac{1}{\omega_n^2 - \omega_a^2} \right)^2$$ [20%]

Examiner's comment:

Parts (a) and (b), involving the derivation of transfer functions and spectra were generally well done. Parts (c) to (e), involving integration of the spectra to yield rms values, were less well done. The students were intended to employ various white noise approximations, and the solution to part (c) was given on the paper to help them spot the appropriate approach. The question will be used as an example on the course next year to ensure that future students have a better understanding of white noise approximations.

Q1

(a)



(b) D.F. (for $\alpha > b$)

$$= \frac{1}{\pi\alpha}\int_0^{2\pi} (\text{output}) \cos\theta \, d\theta$$

$$= \frac{4}{\pi\alpha}\left[\int_0^{\beta} sb\cos\theta\, d\theta + \int_{\beta}^{\pi/2} s\alpha\cos^2\theta\, d\theta\right]$$

$$= \frac{4s}{\pi\alpha}\left[b\sin\beta + \frac{\alpha}{2}\left(\frac{\pi}{2} - \beta - \frac{\sin 2\beta}{2}\right)\right]$$

$$= \frac{2s}{\pi}\left[\left(\frac{\pi}{2} - \beta\right) + \frac{\sin 2\beta}{2}\right]$$

$$= \frac{2s}{\pi}\left[\left(\frac{\pi}{2} - \cos^{-1}\left(\frac{b}{\alpha}\right)\right) + \frac{b}{\alpha}\sqrt{1 - \frac{b^2}{\alpha^2}}\right].$$

(c) For $\alpha < b$, $D = 1$ and system is linear

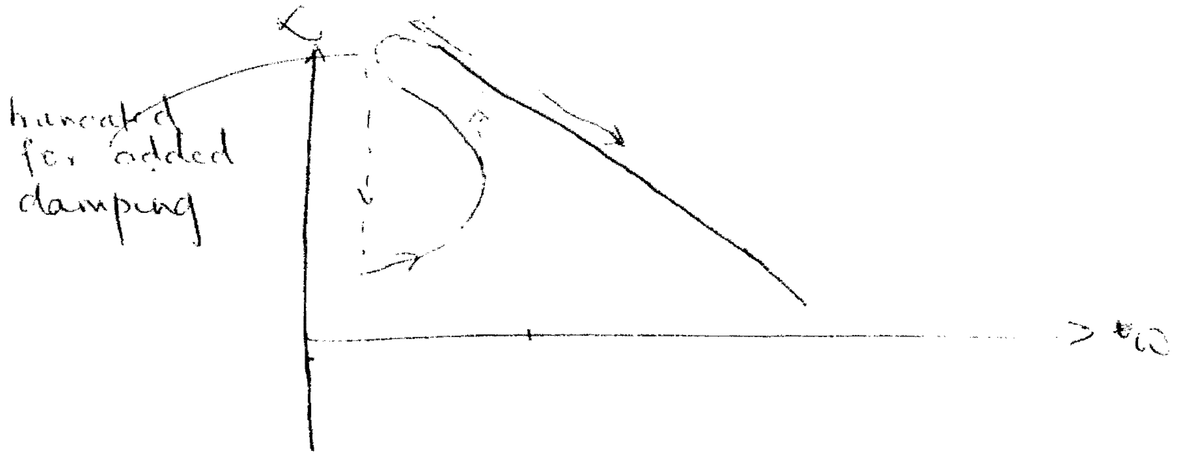
For $\alpha > b$:

$$m\ddot{x} + k(x) = f\cos\omega t$$

$$-m\omega^2\alpha + \frac{2s\alpha}{\pi}\left[\frac{\pi}{2} - \cos^{-1}\left(\frac{b}{\alpha}\right) + \left(\frac{b}{\alpha}\right)\sqrt{1 - \frac{b^2}{\alpha^2}}\right] \simeq f.$$

The above equation relates f, α, ω and can be solved for particular values of f, α.

(d) The system describes a softening spring for $\alpha > b$ The frequency response will keel to the left of the response expected for the linear system. A hysteresis will be observed for scanning forwards and backwards in frequency as anticipated for a system describing "spring softening".



Examiner's comment:

A popular question with most students showing a good understanding of the topic. Most students were able to derive the describing function in part (b), with relatively few students making significant algebraic errors. Part (d) was less well done, with few students giving a fully comprehensive description of the frequency response function.

Q4

(a)

$$\ddot{x} + 2\zeta\dot{x} - \alpha x + x^3 = 0$$

$$y = \dot{x}$$

$$\dot{y} = -2\zeta y + \alpha x - x^3$$

singular points occur when:

$$y = \dot{x} = 0 \quad \text{and}$$

$$\dot{y} = -2\zeta y + \alpha x - x^3 = 0$$

or $\dot{x} = 0$ and $\alpha x - x^3 = 0$

$\dot{x} = 0, x = 0$ and $\dot{x} = 0$, $x = \pm\sqrt{\alpha}$ if $\alpha > 0$

(b)

$\dot{x} = 0, x = 0$:→

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \alpha & -2\zeta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\lambda = -\zeta \pm \sqrt{\zeta^2 + \alpha} \quad \& \ \zeta > 0$$

$\alpha > 0 \Rightarrow$ saddle point

$\alpha < 0 \Rightarrow$ stable node if $\alpha > -\zeta^2$

$\alpha < 0 \Rightarrow$ stable focus if $\alpha < -\zeta^2$

if $\alpha > 0$ then 2 other equilibrium points exist, shifting origin to equilibrium points + linearising)

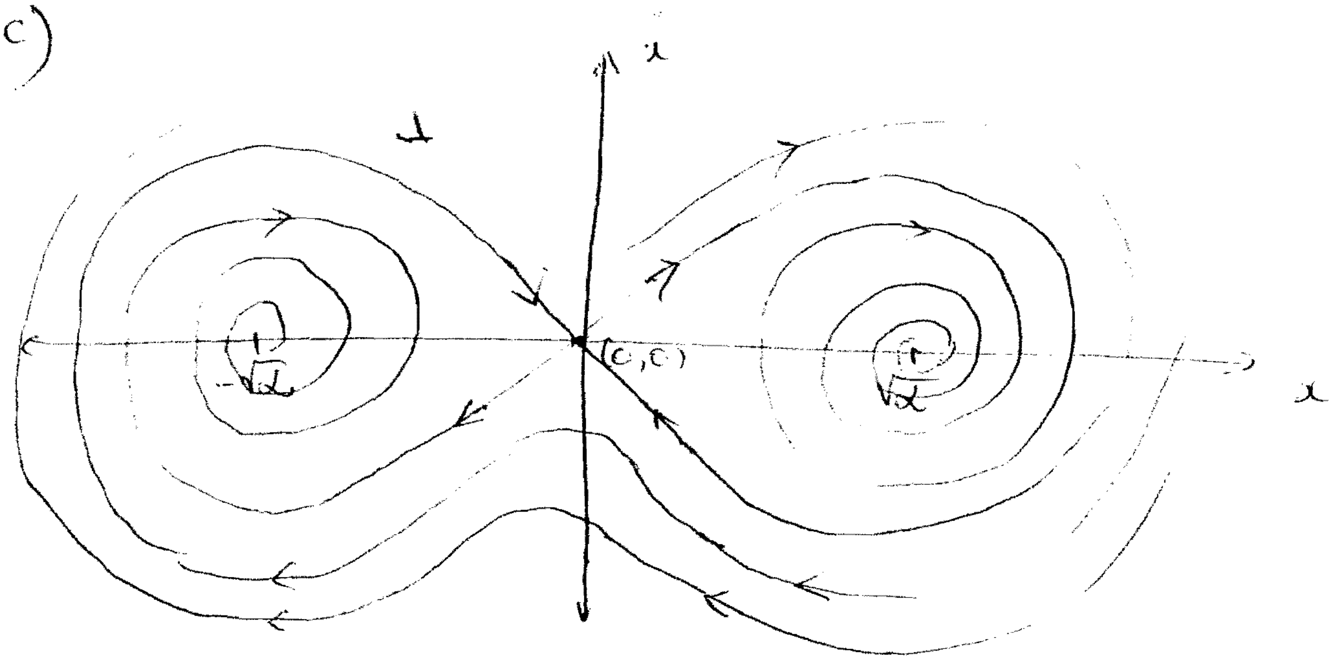
$$\begin{bmatrix} \dot{z} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2\alpha & -2\zeta \end{bmatrix} \begin{bmatrix} z \\ y \end{bmatrix}$$

$$\lambda = -\zeta \pm \sqrt{\zeta^2 - 2\alpha}$$

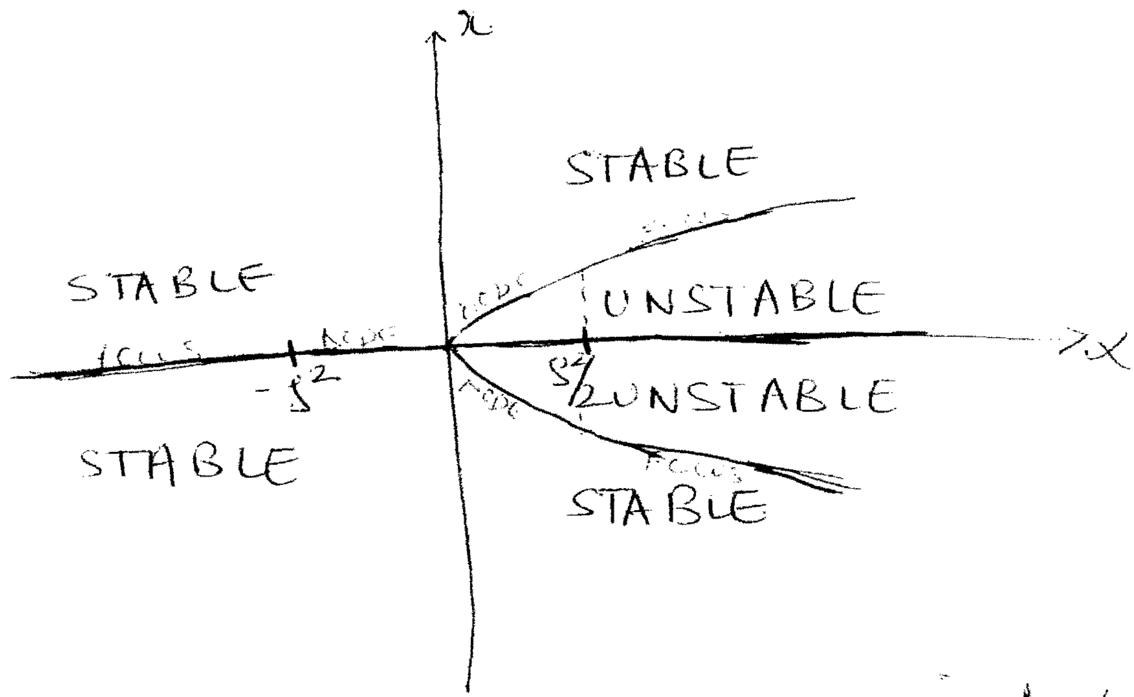
∴ equilibrium point is stable focus if $\alpha > \zeta^2/2$

equilibrium point is stable node if $\alpha < \zeta^2/2$

(c)



(d)



Pitchfork bifurcation is described by the system

Examiner's comments:

Another popular question, with most students being able to identify and classify the singular points of the system in parts (a) and (b). A common error in part (c) was to omit detail from the phase plane diagram. In part (d), the bifurcation diagram was often drawn in error, or drawn without an indication of the significant features.