(a)

(i) Sum on i $e_{ijk}e_{kij} = e_{1jk}e_{k1j} + e_{2jk}e_{k2j} + e_{3jk}e_{k3j}$

next sum on j. The non-zero terms are

$$= e_{12k}e_{k12} + e_{13k}e_{k13} + e_{21k}e_{k21} +$$
$$e_{23k}e_{k23} + e_{31k}e_{k31} + e_{32k}e_{k32}$$

finally sum on k, the non-zero terms are

$$= e_{123}e_{312} + e_{132}e_{213} + e_{213}e_{321} +$$
$$e_{231}e_{123} + e_{312}e_{231} + e_{321}e_{132}$$
$$= 1 + 1 + 1 + 1 + 1 + 1 = 6 \quad (10\%)$$

(ii) $e_{ijk}a_j a_k = e_{i1k}a_1 a_k + e_{i2k}a_2 a_k + e_{i3k}a_3 a_k$

$$= e_{i12}a_1 a_2 + a_{i13}a_1 a_3 + e_{i21}a_2 a_1 +$$
$$e_{i23}a_2 a_3 + a_{i31}a_3 a_1 + e_{i32}a_3 a_2$$

$i = 1$, $e_{1jk}a_j a_k = a_2 a_3 - a_3 a_2 = 0$

$i = 2$, $e_{2jk}a_j a_k = a_1 a_3 - a_3 a_1 = 0$ (10%)

$i = 3$, $e_{3jk}a_j a_k = a_1 a_2 - a_2 a_1 = 0$ $\Rightarrow 0$

~~***~~ Could also say $e_{ijk} a_j a_k = a \times a = 0$!

(b) $\cancel{B_{ik} = e_{ijk} a_j = -(e_{kji} a_j)}$

$$B_{ik} = e_{ijk} a_j = -(e_{kji} a_j) = -B_{ki}$$

(20%)

(c)

(i) $$\sigma_{ij} n_j = \lambda n_i$$

$$\Rightarrow (\sigma_{ij} - \delta_{ij}\lambda) n_j = 0$$

For non-trivial solns. to above homogenous eqns

$$\det(\sigma_{ij} - \lambda \delta_{ij}) = 0$$

(20%)

(ii) Let $\lambda^{(1)}, \lambda^{(2)}$ & $\lambda^{(3)}$ be the principal values.

$$\sigma_{ij} n_j^{(2)} = \lambda^{(2)} n_i^{(2)} \quad \&$$

$$\sigma_{ij} n_j^{(3)} = \lambda^{(3)} n_i^{(3)}$$

Multiplying first by $n_i^{(3)}$ & second by $n_i^{(2)}$

$$\sigma_{ij} n_j^{(2)} n_i^{(3)} = \lambda^{(2)} n_i^{(2)} n_i^{(3)}$$

$$\sigma_{ij} n_j^{(3)} n_i^{(2)} = \lambda^{(3)} n_i^{(3)} n_i^{(2)}$$

Since σ_{ij} is symmetric we can interchange dummy indices i & j (5)

$$\Rightarrow \quad (\lambda^{(2)} - \lambda^{(3)}) n_i^{(2)} n_i^{(3)} = 0$$

Since $\lambda^{(2)} \neq \lambda^{(3)} \Rightarrow n_i^{(2)} n_i^{(3)} = 0$, ie directions (2) & (3) are perpendicular. Same can be done for the other pairs. (40%)

Examiner's comment:

A popular question, well-answered by most candidates. The main part the candidates found difficult was part c(ii) where they had to prove the principal directions are orthogonal- most students did not know how to start going about this.

(2)

(a) $\sigma_{ij} = \sigma_m \delta_{ij} + S_{ij} = \sigma_m^* \delta_{ij} + S_{ij}^*$

with $S_{ii} = S_{ii}^* = 0$. Then

$$\sigma_{ii} = 3\sigma_m = 3\sigma_m^* \Rightarrow \sigma_m = \sigma_m^*$$

& $\sigma_m \cancel{\delta_{ij}} + S_{ij} = \sigma_m \cancel{\delta_{ij}} + S_{ij}^*$

$$\Rightarrow S_{ij} = S_{ij}^*$$

(30%)

(b)

(i) $\phi = \dfrac{Cr^2}{\tan\alpha - \alpha}\left(\alpha - \theta + \dfrac{\sin 2\theta}{2} - \tan\alpha \cos^2\theta\right)$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2} = \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\theta - \sin 2\theta - 2\tan\alpha \sin^2\theta\right]$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} = \frac{C}{\tan\alpha - \alpha}\left[2\alpha - 2\theta + \sin 2\theta - 2\tan\alpha \cos^2\theta\right]$$

(20%)

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right) = \frac{C}{\tan\alpha - \alpha}\left[1 - \cos 2\theta - 2\tan\alpha \sin\theta \cos\theta\right]$$

(c) along OB $\sigma_{r\theta} = \sigma_{\theta\theta} = 0$

(ii) along OA $\sigma_{r\theta} = 0$; $\sigma_{\theta\theta} = -p$

$$\sigma_{\theta\theta}\Big|_{\theta=\alpha} = \frac{C}{\tan\alpha-\alpha}\left[2\alpha - 2\alpha - \sin 2\alpha - 2\sin\alpha\cos\alpha\right] = 0$$

$$\sigma_{r\theta}\Big|_{\theta=\alpha} = \frac{C}{\tan\alpha-\alpha}\left[1-\cos 2\alpha - 2\sin^2\alpha\right] = 0$$

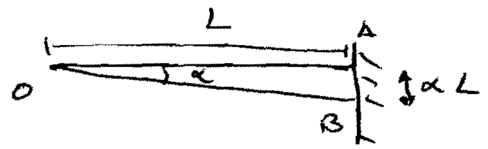
$\Rightarrow$ B.C. along OB satisfied

$$\sigma_{\theta\theta}\Big|_{\theta=0} = \frac{C}{\tan\alpha-\alpha}\left[2\alpha - 2\tan\alpha\right] = -2C = -p$$

$$C = p/2$$

$$\sigma_{r\theta}\Big|_{\theta=0} = \frac{C}{\tan\alpha-\alpha}\left[1-1-0\right] = 0 \qquad (20\%)$$

(c)



considers small α

(ii)

$$M = pL\frac{L}{2} = \frac{pL^2}{2}$$

$$I \text{ at } AB = \frac{bd^3}{12} = \frac{(\alpha L)^3}{12}$$

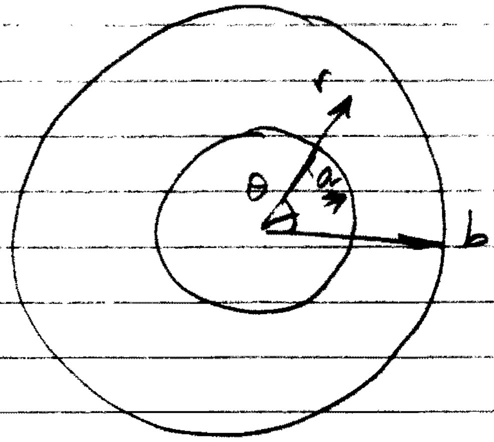
$$\Rightarrow \sigma = \frac{My}{I} = \frac{pL^2}{2}\,\frac{\alpha L}{2}\,\frac{12}{\alpha^3L^3} = \frac{3p}{\alpha^2} \qquad (30\%)$$

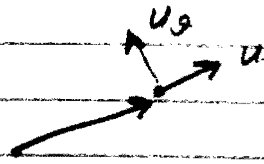
$$\sigma_{rr}\Big|_{\theta=0} = \frac{p}{2(\tan\alpha-\alpha)}(2\alpha)\ ; \quad \tan\alpha \sim \alpha + \frac{\alpha^3}{3} \text{ for small } \alpha$$

$$\Rightarrow \sigma_{rr}\Big|_{\theta=0} = \frac{p}{2\alpha^3} = \frac{3p}{\alpha^2}$$

Examiner's comment: On the whole the question was again well attempted. The main difficulty was in part (a) where they had to prove that the stress tensor could be decomposed into the hydrostatic and deviatoric parts in only one way.

3 (a)





$$u_r = \frac{A}{r} \qquad u_\theta = u_z = 0 \qquad A > 0$$

$$\dot{\varepsilon}_{rr} = \frac{\partial u_r}{\partial r} = \frac{-A}{r^2} \qquad \dot{\varepsilon}_{\theta\theta} = \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{u_r}{r}$$

$$\Rightarrow \quad \dot{\varepsilon}_{\theta\theta} = A / r^2$$

(35%)

$$\dot{\varepsilon}_{ii} = \dot{\varepsilon}_{rr} + \dot{\varepsilon}_{\theta\theta} + \dot{\varepsilon}_{zz} = 0$$

$$\dot{\varepsilon}_{zz} = \frac{\partial u_z}{\partial z}$$

(b) $$\dot{\varepsilon}_{ij} = \frac{3}{2}\frac{s_{ij}}{\sigma_Y}\frac{\dot{\sigma}_e}{h} \qquad \frac{\dot{\sigma}_e}{h} = \dot{\varepsilon}_e$$

$$\dot{w} = \sigma_{ij}\dot{\varepsilon}_{ij} = s_{ij}\dot{\varepsilon}_{ij} = \left(\frac{3}{2}s_{ij}s_{ij}\right)\frac{\dot{\varepsilon}_e}{\sigma_Y}$$

Now, $\sigma_Y^2 = \frac{3}{2}s_{ij}s_{ij} \Rightarrow \dot{w} = \sigma_{ij}\dot{\varepsilon}_{ij} = \sigma_Y\dot{\varepsilon}_e$

Now $\dot{\varepsilon}_e^2 = \frac{2}{3}\dot{\varepsilon}_{ij}\dot{\varepsilon}_{ij} = \frac{2}{3}\left(\dot{\varepsilon}_{rr}^2 + \dot{\varepsilon}_{\theta\theta}^2\right)$

$$= \frac{2}{3}\frac{2A^2}{r^4} \Rightarrow \dot{\varepsilon}_e = \frac{2}{\sqrt{3}}\frac{A}{r^2}$$

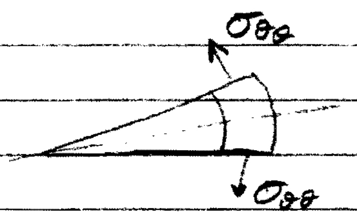
$$p \cdot 2\pi a\, u_r(r=a) = \int_a^b \dot{w}(r) \cdot 2\pi r\, dr$$

$$\Rightarrow \quad p \cdot 2\pi \cancel{a}\frac{A}{\cancel{a}} = \int_a^b \frac{2}{\sqrt{3}}\frac{A}{r^2}\sigma_Y 2\pi r\, dr$$

$$\Rightarrow \quad p = \frac{2}{\sqrt{3}} \sigma_Y \int_a^b \frac{1}{r}\, dr = \frac{2}{\sqrt{3}} \sigma_Y \ln \frac{b}{a}$$

(30%)

Equilib^{m}. $\quad \dfrac{\partial}{\partial r}(r\,\sigma_{rr}) = \sigma_{\theta\theta}$



$$\Rightarrow \quad r\frac{\partial \sigma_{rr}}{\partial r} = \sigma_{\theta\theta} - \sigma_{rr}$$

$$= S_{\theta\theta} - S_{rr}$$

Now : $\quad \dot{\varepsilon}_{ij} = \dfrac{3}{2}\dfrac{S_{ij}}{\sigma_Y}\dot{\varepsilon}_e \quad$ via flow rule.

$$\Rightarrow \quad \frac{S_{ij}}{\sigma_Y} = \frac{2}{3}\frac{\dot{\varepsilon}_{ij}}{\dot{\varepsilon}_e}$$

$$\Rightarrow \quad \frac{S_{rr}}{\sigma_Y} = \frac{2}{3}\frac{\dot{\varepsilon}_{rr}}{\dot{\varepsilon}_e} = \frac{\cancel{2}}{3}\left(-\frac{\cancel{A}}{\cancel{r^2}}\right)\cdot\frac{\sqrt{3}\,\cancel{r^2}}{\cancel{2A}}$$

$$= -\frac{1}{\sqrt{3}} \quad \Rightarrow \quad S_{rr} = -\frac{1}{\sqrt{3}}\sigma_Y$$

Also, $\quad S_{\theta\theta} = \dfrac{1}{\sqrt{3}}\sigma_Y \quad$ and $\quad S_{zz} = 0$

$$\Rightarrow \quad r\frac{\partial \sigma_{rr}}{\partial r} = S_{\theta\theta} - S_{rr} = \frac{2}{\sqrt{3}}\sigma_Y$$

$$\Rightarrow \quad \int_{-p}^{\sigma_{rr}} d\sigma_{rr} = \int_a^r \frac{2}{\sqrt{3}}\frac{\sigma_Y}{r}\, dr = \frac{2}{\sqrt{3}}\sigma_Y \ln\frac{r}{a}$$

$$\Rightarrow \quad \sigma_{rr} + p = \frac{2}{\sqrt{3}}\sigma_Y \ln\frac{r}{a}$$

Now at $r = a \Rightarrow \sigma_{rr} = -p$ ✓

at $r = b \Rightarrow \sigma_{rr} = \underbrace{\left(\frac{2}{\sqrt{3}}\sigma_Y \ln\frac{b}{a}\right) - p}_{=0}$ ✓

So equilib^{m} is satisfied.

Hydostatic stress $\sigma_h(r) = ?$

$\sigma_{rr} = \sigma_h + S_{rr} \Rightarrow \sigma_h = \sigma_{rr} - S_{rr}$

$$\Rightarrow \sigma_h = \left(\frac{2}{\sqrt{3}}\sigma_Y \ln\frac{r}{a}\right) + \frac{1}{\sqrt{3}}\sigma_Y$$

(35%)

Examiner's comment:

The least popular question but on the whole poorly attempted. Students managed to derive the strain rates from the given velocity field but failed to do the remaining two parts of the questions that involved deriving an upper bound and then showing that it was the exact solution.