Part IIB 2012 4C15 MEMS Design

1(a) $\zeta = b/2\sqrt{km} = 10^{-4}/2\sqrt{10\times10^{-7}} = 0.05$ Mechanics Data Book

$f_r = f_n\sqrt{1-2\zeta^2}$

$= \frac{1}{2\pi}\sqrt{\frac{10}{10^{-7}}}\sqrt{1-2(.05)^2}$

$= 1.59$ kHz $\omega_r = 9.97\,s^{-1}$

$\Delta x = \frac{a}{\omega_r^2} = \frac{9.81}{(9.97)^2} = 98.6\times10^{-9}$ m $= 98.6$ nm

ie. 1g

(b) $C = N \times C_u$ N = no of unit cells
C_u = capacitance of each cell

$C_u = \frac{8.85\times10^{-12}\times10\times750\times10^{-6}}{2} = 33.1\times10^{-15}$ F

$N = 2\times\frac{2.5\times10^{-3}}{44\times10^{-6}} \Rightarrow 112$

2 sides

$\therefore C = 112\times33.1\times10^{-15} \Rightarrow 3.77$ pF

(c) $\frac{\Delta C}{C} = \frac{2x}{g\ (\text{gap})} = \frac{2\times98.6\times10^{-9}}{2\times10^{-6}} = 98.6\times10^{-3}$

ie. 9.86%

(d) $\overline{F_n} = \sqrt{4k_BT\times b} = \sqrt{4\times1.38\times10^{-23}\times300\times10^{-4}}$

$\Rightarrow 1.287\times10^{-12}$ N/$\sqrt{\text{Hz}}$

$\therefore \frac{\overline{F_n}}{k} = \overline{x}_n = \frac{1.287\times10^{-12}}{10} = 1.29\times10^{-13}$ m/$\sqrt{\text{Hz}}$

$\therefore \overline{a}_n = \frac{1.29\times10^{-13}}{98.6\times10^{-9}}$ g/$\sqrt{\text{Hz}}$ $= 1.31\times10^{-6}$ g/$\sqrt{\text{Hz}}$

from (a)

or 1.28×10^{-5} m s^{-2}/$\sqrt{\text{Hz}}$

(e) It is usually assumed that increasing the proof mass area &/or structural thickness and reducing the gap g improves the noise equivalent resolution. However to first order, sensitivity is independent of structural thickness and gap and tuning these parameters may actually increase the fluidic damping and hence the thermo-mechanical noise equivalent acceleration.

Tuning these parameters in the opposite direction will result in lower capacitance and capacitance sensitivity that may degrade the electronic noise equivalent resolution.

Increasing the overall area may be beneficial in increasing mechanical sensitivity - by reducing the natural frequency and also allowing for increased capacitance thereby reducing the electronic noise equivalent rate. However this could also lead to increased fluidic damping via Couette drag against the substrate. A process level option to reduce such fluidic dissipation is vacuum packaging.

Examiners comments

Generally well done and mistakes more often numeric or algebraic rather than conceptual. Rather concerning was the fact that several candidates forgot about 2π when specifying frequency in Hz.

2. (a)

$$F = \underbrace{\frac{\varepsilon_0 t V^2}{2g_0}}_{\text{comb drive}} + \underbrace{\frac{\varepsilon_0 A V^2}{2(g_0 - x)^2}}_{\text{//el plate}}$$

for derivation see Lecture notes

(b) $$k_e = \frac{\partial F}{\partial x} = \frac{-2.-1\,\varepsilon_0 A V^2}{2(g_0 - x)^3} = \frac{\varepsilon_0 A V^2}{(g_0 - x)^3}$$

(c) at pull-in forces balance $$kx = \frac{\varepsilon_0 t V^2}{2g_0} + \frac{\varepsilon_0 A V^2}{2(g_0 - x)^2} \quad \text{— ①}$$

But also $k_m = k_e = k$

ie. $$\frac{\varepsilon_0 A V^2}{(g_0 - x)^3} = k \quad \text{— ②}$$

Substituting for k in ①

$$\frac{\varepsilon_0 A V^2 . x}{(g_0 - x)^3} = \frac{\varepsilon_0 t V^2}{2g_0} + \frac{\varepsilon_0 A V^2}{2(g_0 - x)^2}$$

ie. $$2 g_0 x A = t(g_0 - x)^3 + A g_0 (g_0 - x)$$

Now put $z = g_0 - x$ ie. $x = g_0 - z$

$$2 g_0 A (g_0 - z) = t z^3 + A g_0 z$$

ie. $$z^3 + \frac{3 A g_0}{t} z = \frac{2 g_0^2 A}{t}$$

So using provided hint

$$z_{PI} = \left\{\frac{g_0^2 A}{t} + \sqrt{\frac{g_0^3 A^3}{t^3} + \frac{g_0^4 A^2}{t^2}}\right\}^{1/3} + \left\{\frac{g_0^2 A}{t} - \sqrt{\frac{g_0^3 A^3}{t^3} + \frac{g_0^4 A^2}{t^2}}\right\}^{1/3}$$

writing as $z_{PI} = (a + \sqrt{b})^{1/3} + (a - \sqrt{b})^{1/3}$

$a = \frac{g_0^2 A}{t}$ $\sqrt{b} = \sqrt{\;---}$

$$z_{PI}^3 = (a+\sqrt{b}) + 3(a+\sqrt{b})^{2/3}(a-\sqrt{b})^{1/3} + 3(a+\sqrt{b})^{1/3}(a-\sqrt{b})^{2/3} + (a-\sqrt{b})$$

$$= 2a + 3(a+\sqrt{b})^{1/3}(a-\sqrt{b})^{1/3}\left\{(a+\sqrt{b})^{1/3} + (a-\sqrt{b})^{1/3}\right\}$$

ie. $z_{PI}^3 \Rightarrow 2a + 3(a^2-b)^{1/3} \cdot z_{PI}$

But from ② $z_{PI}^3 = \dfrac{\varepsilon_0 A V_{PI}^2}{k}$

$$\therefore \frac{\varepsilon_0 A V_{PI}^2}{k} = 2a + 3(a^2-b)^{1/3} z_{PI}$$

Now substitute for a & b, $a = \dfrac{g_0^2 A}{t}$; $b = \dfrac{g_0^3 A^3}{t^3} + \dfrac{g_0^4 A^2}{t^2}$

$$\therefore \frac{\varepsilon_0 A V_{PI}^2}{k} = \frac{2g_0^2 A}{t} + 3\left\{\cancel{\frac{g_0^4 A^2}{t^2}} - \frac{g_0^3 A^3}{t^3} - \cancel{\frac{g_0^4 A^2}{t^2}}\right\}^{1/3} z_{PI}$$

$$\therefore \frac{\varepsilon_0 A V_{PI}^2}{k} = \frac{2g_0^2 A}{t} - \frac{3 g_0 A}{t}$$

$$\therefore V_{PI}^2 = \frac{2kg_0^2}{\varepsilon_0 t}\left\{1 - \frac{3 z_{PI}}{g_0}\right\}$$

(d) System provides advantages of both comb drive and parallel plate actuators. The comb drive allows for large displacements with relatively linear response while the force generated from a parallel plate actuator is highly non-linear the force density, ie. force per unit area, is much larger than a comb drive. A design compromise is therefore possible by trading relative areas.

However, the system is still limited by the 'pull-in instability' for voltage controlled operation - so approximates to a parallel plate actuator with improved linearity and added range.

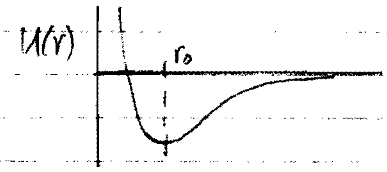
<u>Examiner's comments</u> candidates invariably quoted rather than derived the expressions in (a). Most candidates got to the cubic if not its solution & so gained reasonable credit for (c).

3 (a) The force between two atoms has two components. a short range repulsive force and a longer range attractive component.

When two atoms are brought together their electron clouds overlap – as this is energetically unfavourable a repulsive force is generated arising from the Pauli exclusion principle. The essential characteristic is that its magnitude rises very steeply as separation distance r falls. Typically the interaction energy varies with r^{-12}.

The force of attraction is known generically as the van der Waals force and can arise from a number of sources. Its potential too falls off with distance r but less steeply, typically with r^{-6}

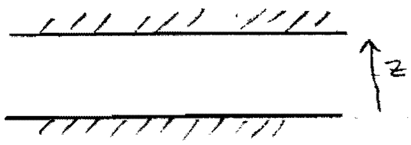
Hence potential $U(r) = -\frac{C}{r^6} + \frac{D}{r^{12}}$



For a fuller discussion of the origin of these terms see the relevant sections of:

- Israelachvili J 'Intermolecular & Surface Forces' LG287
- Mate CM 'Tribology on the small scale' RP182
- Maugis D 'Contact Adhesion & Rupture of Elastic Solids' JJ330

(b) Considering only attractive or non-retarded term and integrating for two parallel planes of atoms provides the quoted result. This was first done by Hamaker about 80 yrs ago.



$$U(z) = -\frac{A}{12\pi z^2}$$

But $F = -\frac{dU(z)}{dz} = \frac{A}{6\pi z^3}$

Consider unit area

But $w = \int_{h_0}^{\infty} F\,dz = \frac{A}{6\pi}\int_{h_0}^{\infty} z^{-3}\,dz = \left[\frac{-A}{12\pi z^2}\right]_{h_0}^{\infty}$

$$= \frac{A}{12\pi h_0^2}$$

note that since $F = -\frac{dU(z)}{dz}$

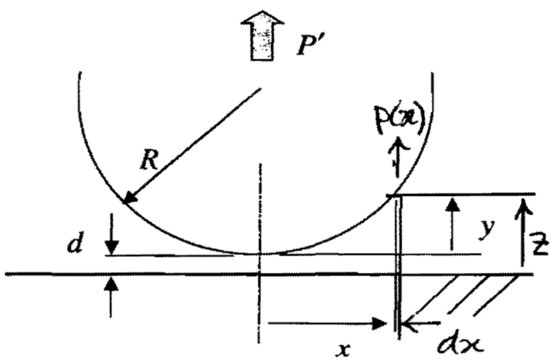
and $w = \int_{h_0}^{\infty} F\,dz \quad w = -\int_{h_0}^{\infty} dU(z)$

$= -U(h_0) \quad = \frac{A}{12\pi h_0^2}$

for any function of $U(z)$

(c) The Derjaguin approximation is a geometric argument which allows the force law between two curved surfaces to be expressed in terms of the interaction energy between two planar surfaces. The demonstration of this ~~relies~~ relies on the fact that the radius (or radii) of curvature of the convex body are large compared to the separation of the surfaces and thus at a given distance from the point of nearest approach can be in essence parallel.

(d)



If $y = \frac{x^2}{2R}$

'usual' parabolic approx to circle

$dP' = p(z)\,dx \times 2$

(2 sides) per unit leng

But $z = d + y$

$\therefore dz = dy = \frac{2x}{2R}\,dx$

ie. $dz = \frac{x}{R}\,dx$

$\therefore dP' = 2 \times \frac{R}{x}\, p(z)\,dz$

But $x = \sqrt{2R(z-d)}$

Now for plane surfaces $U(z) = -\frac{A}{12\pi z^2}$

ie. $U(z) = -w h_0^2 z^{-2}$

so treating <u>axial strip</u> as being ∥el to contaface

$$p(z) = -\frac{\partial U(z)}{\partial z} = \frac{2wh_0^2}{z^3}$$

$$\therefore \quad dP' = 2 \cdot \frac{2wh_0^2}{z^3} \cdot \frac{R}{\sqrt{2R(z-d)}}\, dz$$

per unit length $\rightarrow$ $$P' = 2wh_0^2\sqrt{2R}\int_d^\infty \frac{dz}{z^3(z-d)^{1/2}}$$

Consider $I = \int_d^\infty \frac{dz}{z^3(z-d)^{1/2}}$

if $z \gg d$ then $I \simeq \int_d^\infty \frac{dz}{z^{7/2}} = -\left[\frac{2}{5} z^{-5/2}\right]_d^\infty = \frac{2}{5d^{5/2}}$

hence $$P' = 2wh_0^2\sqrt{2R} \cdot \frac{2}{5d^{5/2}}$$

$$P' = \frac{4wh_0^2}{5}\sqrt{\frac{2R}{d^5}}$$

Quite possible to find analytical solution to $I = \int_d^\infty \frac{dz}{z^3(z-d)^{1/2}}$ by substitution $z-d=y^2$ then $d=a^2$, then $\theta = \tan^{-1} y/a$

whence $$I = \frac{2}{d^{5/2}}\int_0^{\pi/2} \cos^4\theta\, d\theta = \frac{3\pi}{16}$$

hence $$P' = wh_0^2\sqrt{\frac{2R}{d^5}} \cdot \frac{3\pi}{8}$$

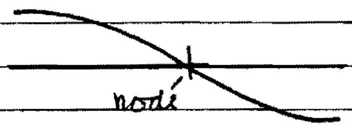
1.17

(e) In addition to problems of surface topography & cleanliness, experimentally maintaining alignment for any nominal line contact would be prohibitively difficult. All such experiments use point contacts - sphere on plane or crossed cylinders

4 (a) Q is high so that $\omega_n = \sqrt{K/m}$

or $f_n = \frac{1}{2\pi}\sqrt{\frac{\pi^2}{8}\frac{Ewh}{L} \times \frac{2}{\rho w h L}} = \frac{1}{4L}\sqrt{\frac{E}{\rho}}$

$\therefore f_n = \frac{1}{4\times 180\times 10^{-6}}\sqrt{\frac{160\times 10^{9}}{2330}} = \underline{11.5\ \text{MHz}}$



(b) ‖el plate actuator $F_{\parallel} = \frac{\varepsilon_0 A V_{DC} V_{AC}}{g^2}$

amplitude at resonance, $X \simeq \frac{Q.F_{\parallel}}{k}$

$X = Q\,\frac{\varepsilon_0 A V_{DC} V_{AC}}{g^2}\,\frac{8}{\pi^2}\,\frac{L}{Ewh}$

$= \frac{10^5 \times 8.854\times 10^{-12} \times 10 \times 8 \times 10^{-12} \times 50 \times 1}{(1\times 10^{-6})^2} \times \frac{8}{\pi^2} \times \frac{180\times 10^{-6}}{160\times 10^{9} \times 10 \times 8\times 10^{-12}}$

$= 40\,4 \times 10^{-9}\ \text{m}$

i.e. $\underline{40.4\ \text{nm}}$

i_m motional current $= \frac{dQ}{dt}$

from (a) from (b)

$= V_P\frac{dC}{dt} = V_P\frac{d}{dt}\left(\frac{\varepsilon_0 A}{g-x}\right) = \frac{V_P\varepsilon_0 A}{(g-x)^2}\,\omega X$

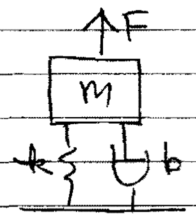
$g - x$ = gap when displacement x, so when $x = 0$

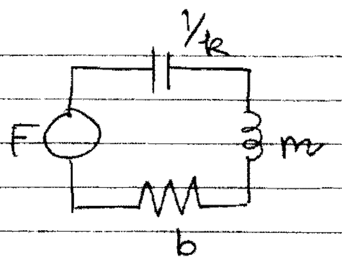
$i_m = \frac{50\times 8.85\times 10^{-12}\times 10\times 8\times 10^{-12}}{1\times 1\times 10^{-12}} \times 11.5\times 10^{6}\times 2\pi \times 40.4\times 10^{-9}$

$= 1.03\times 10^{-7}$ amp

$= \underline{0.103\ \mu\text{amp}}$

(c) The required expressions can be written down from analogy between a mechanical and an electrical system





$$k = \frac{\pi^2 E w h}{8\, L}$$

$$m = \frac{\rho w L h}{2}$$

$$R_m = \frac{b}{\eta^2}, \quad L_m = \frac{m}{\eta^2}, \quad C_m = \frac{\eta^2}{k}$$

$$\eta = V_p \frac{dC}{dx} = \frac{V_p \varepsilon_0 A}{g^2} \quad \text{for asymmetric } /\!/\text{el plate}$$

$$\eta = \frac{100 \times 8.854 \times 10^{-12} \times 10 \times 8 \times 10^{-12}}{(1 \times 10^{-6})^2} = 7.08 \times 10^{-8}$$

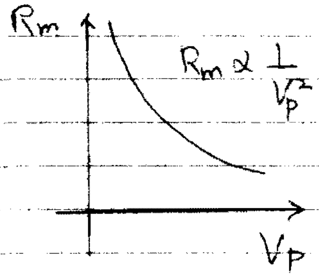
$$\therefore C_m = \frac{(7.08 \times 10^{-8})^2 \times 8 \times 180\ 10^{-6}}{\pi^2 \times 160 \times 10^9 \times 10 \times 8 \times 10^{-12}} \Rightarrow \underline{57.1 \times 10^{-21}\ F}$$

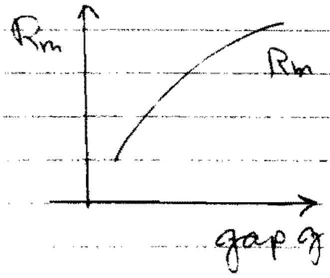
$$L_m = \frac{2330 \times 10 \times 180 \times 8 \times 10^{-18}}{2 \times (7.08 \times 10^{-8})^2} \Rightarrow \underline{3347\ H.}$$

$$R_m = \frac{m\omega_r}{Q \eta^2} = \frac{2330 \times 10 \times 180 \times 8 \times 10^{-18} \times 11.5 \times 10^6 \times 2\pi}{2 \times 10^5 \times (7.08 \times 10^{-8})^2}$$

$$= 2.42\ M\Omega$$

(d)





Motional resistance may be minimised by reducing gap g – however, this reduces the motional amplitude and hastens the onset of non-linearities – or increasing the DC voltage – this usually has a limit either due to nature of process or by pull-in instability. Increasing area through scaling thickness provides some benefit – alternative resonator topologies are also possible.

Examiner's comments

3. Parts (a) & (b) well done – explanations for Derjaguin approximation were sometimes a bit confused involving – erroneously – elasticity and condensation bridges. Elasticity of surfaces is accounted for in either JKR or DMT treatments. The difficulty about evaluating the integral in (d) was appreciated by some candidates.

4. Several candidates either didn't bother about mode shape – or else put nodes at the ends of the free-free beam and an antinode at the centre.

JAW
April 12

2012 4C15 Answers

1 (a) 1.59 kHz 98.6 nm (b) 3.77 pF (c) 9.9% (d) $1.28 \times 10^{-5}\ \mathrm{ms^{-2}}/\sqrt{\mathrm{Hz}}$

2. (a) $\dfrac{\varepsilon_0 t V^2}{2g_0}$, $\dfrac{\varepsilon_0 A V^2}{2(g_0 - x)^2}$ (b) $\dfrac{\varepsilon_0 A V^2}{(g_0 - x)^3}$ (c) $V_{PI}^2 = \dfrac{2kg_0^2}{\varepsilon_0 t}\left\{1 - \dfrac{3z_{PI}}{g_0}\right\}$; $z_{PI}^3 = \dfrac{\varepsilon_0 V_{PI}^2}{k}$

3 $P' = \dfrac{3\pi}{8} w h_0^2 \sqrt{\dfrac{2R}{d^5}}$

4 (a) 11.5 MHz, (b) 40.4 nm 0.103 μamp (c) $R_m = 2.42\,\mathrm{M\Omega}$, $L_m = 3347\,\mathrm{H}$, $C_m = 57.1 \times 10^{-21}\,\mathrm{F}$