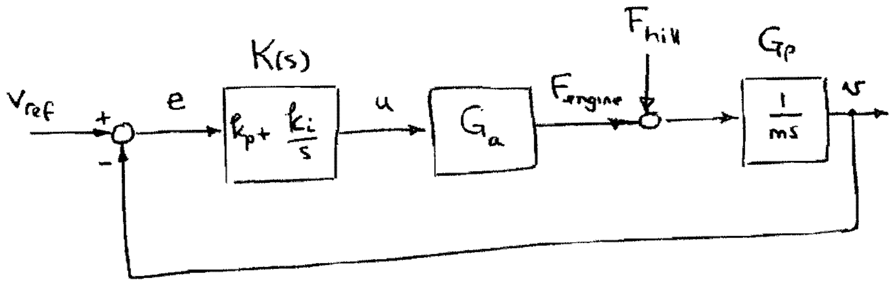
Engineering Tripos Part IIB, Module 4F1,
CONTROL SYSTEM DESIGN
SAMPLE SOLUTIONS TO EXAM MAY 2012

1. **Solution:**

(a) Bookwork on two-degree-of-freedom control systems. The resulting closed-loop system from reference to output has to contain the same RHP zeros as the plant, and the relative degree of the closed loop must be at least as high as the plant.



(b) i.

ii. Loop gain, $L(s) = \frac{(sk_p+k_i)}{s}\frac{B}{(s+\alpha)}\frac{1}{ms}$, and the closed loop poles satisfy $1 + L(s) = 0$ giving the characteristic equation:

$$s^2(s+\alpha) + \frac{B}{m}(sk_p + k_i) = 0$$

The roots will all be stable by the Routh-Hurwitz condition for a third order polynomial, requiring all the coefficients being positive (i.e. $\alpha > 0$, $B > 0$, $k_p > 0$, $k_i > 0$), and

$$\alpha\frac{B}{m}k_p > \frac{B}{m}k_i \Leftrightarrow \boxed{\alpha > \frac{k_i}{k_p}}$$

iii.

$$\frac{\hat{e}}{\hat{v}_{ref}} = \frac{1}{1 + K(s)G_a(s)G_p(s)} = \frac{s^2(s+\alpha)}{s^2(s+\alpha) + \frac{B}{m}(sk_p + k_i)} = S(s)$$

which clearly has a double zero at $s = 0$.

iv.

$$\hat{v}_{ref} = \frac{1}{s} \Rightarrow \hat{e} = S(s)\frac{1}{s} = \frac{s(s+\alpha)}{s^2(s+\alpha) + \frac{B}{m}(sk_p + k_i)} \Rightarrow \hat{e}(0) = 0$$

But

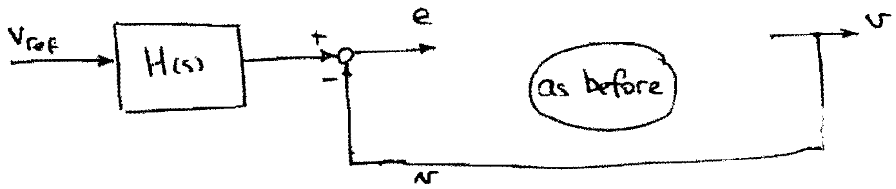
$$\hat{e}(s) = \int_0^\infty e(t)e^{-st}\,dt \Rightarrow \hat{e}(0) = \int_0^\infty e(t)e^0\,dt = \int_0^\infty e(t)\,dt = 0$$

Also $e(0) = v_{ref}(0) - v(0) = 1$, so that we must have $v(t) > v_{ref}(t)$ for a range of t to get $\int_0^\infty (v_{ref}(t) - v(t))\,dt = 0$. i.e. $v(t)$ must overshoot $v_{ref}(t) = 1$.

v. The step response of $R(s) = \frac{2}{(s+1)(s+2)}$ is given by

$$Y(s) = R(s)\frac{1}{s} = \frac{2}{s(s+1)(s+2)} = \frac{1}{s} - \frac{2}{s+1} + \frac{1}{s+2}$$

$\Rightarrow y(t) = 1 - 2e^{-t} + e^{-2t} < 1$ for all $t > 0$ since $2e^{-t} > e^{-2t}$ for all $t > 0$



Want $H(s)\frac{KG_aG_p}{1+KG_aG_p} = R(s)$

$$\Rightarrow H(s) = \frac{\left(s^2(s+\alpha) + \frac{B}{m}(sk_p + k_i)\right)2}{\frac{B}{m}(sk_p + k_i)(s+1)(s+2)}$$

Note that the degree of the numerator is equal to the degree of the denominator so this is OK.

vi.

$$\hat{v} = \frac{G_p}{1+KG_pG_a}\hat{F}_{hill} = \frac{\frac{1}{m}s(s+\alpha)}{s^2(s+\alpha) + \frac{B}{m}(sk_p + k_i)}\hat{F}_{hill}$$

Step change in the force due to gravity on the hill gives: $F_{hill}(t) = -mg\sin(\theta)$ for $t > 0$. The zero steady state gain will mean that the hill has no steady state effect on the velocity and $v(t) \to v_0$. The initial slope is given by the initial value theorem,

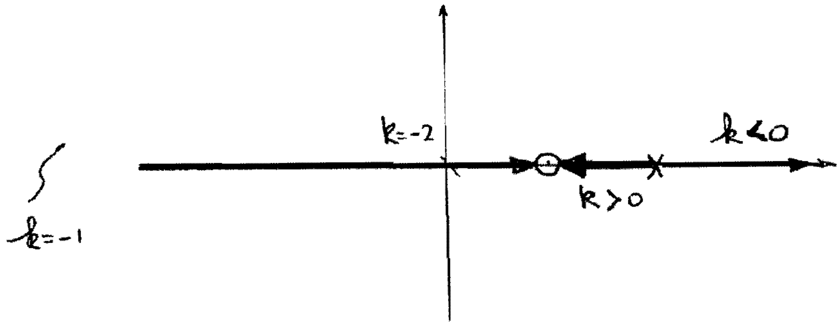
$$\dot{v}(0) = \lim_{s\to\infty} s(s\hat{v}(s)) = \lim_{s\to\infty} s^2\frac{G_p}{1+KG_aG_p} \times \frac{-mg\sin(\theta)}{s} = -g\sin(\theta)$$

2. **Solution:**

(a) i. $G(s) = \frac{s-2}{s-1}$. Closed-loop poles given by

$$1 + kG(s) = 0 \Rightarrow s - 2 + k(s-1) = 0 \Rightarrow s = \frac{k+2}{k+1}$$

and the gain is stabilizing if $\boxed{-2 < k < -1}$. Root locus diagram is:



ii. As the gain is increased in magnitude ($k < 0$) the closed-loop pole appears to pass through $+\infty$ and re-appear at $-\infty$. This makes the closed loop very sensitive to infinitesimal delays. e.g. if $k = -1.5$ when $kG(0) = -0.75$ and $kG(j\infty) = -1.5$ then the Nyquist diagram will be a circle encircling the -1 point once in an anticlockwise direction implying stability since there is one RHP pole in G. However multiplying G by $e^{-\tau s}$ will give more encirclements of the -1 point due to the high frequency gain being > 1 in magnitude, and hence instability for any arbitrarily small time delay $\tau > 0$.

iii. Advice A: do not use a constant gain controller. However even with a dynamic controller the stability margins will be very poor due to the RHP zero being at a lower frequency that the RHP pole.

Advice B: Changing the unstable pole may not be possible since it may be an inherent feature of the system to be controlled, however taking a different measurement or additional measurements could change the position of the RHP zero.

(b) $G(s) = \dfrac{1}{s-2}$ and we require

A: $|T(j\omega)| \leq 2$ for all ω.

B: $|T(j\omega)| \leq \epsilon$ for all $\omega \geq 1$.

i. With controller, $K(s)$, and loop gain, $L(s) = K(s)G(s)$, then $T(s) = \dfrac{L}{1+L} = \dfrac{KG}{1+KG} = \dfrac{K(s)}{s-2+K(s)}$. For internal stability we have $K(2) \neq 0$ (i.e. K cannot have a zero cancelling the unstable pole in G). Hence

$$\boxed{T(2) = \frac{K(2)}{0+K(2)} = 1} \quad \text{independent of the controller.}$$

ii. Let $K(s) = n(s)/d(s)$ with the roots of $n(s)$ in the LHP. Then $T(s) = n(s)/((s-2)d(s)+n(s))$ will have no poles or zeros in $\text{Re}(s) > 0$, and hence the Poisson integral can be used (as given in the data sheet). Hence if A and B are satisfied,

$$\begin{aligned}
0 &= \log(1) = \log(T(2)) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{2}{4+\omega^2}\log(T(j\omega))\, d\omega \\
\Rightarrow 0 &= \int_0^{\infty} \frac{2}{4+\omega^2}\log|T(j\omega|\, d\omega \\
&= \int_0^{1} \frac{2}{4+\omega^2}\log|T(j\omega|\, d\omega + \int_1^{\infty} \frac{2}{4+\omega^2}\log|T(j\omega|\, d\omega \\
&\leq \int_0^{1} \frac{2}{4+\omega^2}\log(2)\, d\omega + \int_1^{\infty} \frac{2}{4+\omega^2}\log(\epsilon)\, d\omega \\
&= [\tan^{-1}(1/2) - 0]\log(2) + [\pi/2 - \tan^{-1}(1/2)]\log(\epsilon) \\
\Rightarrow \log(\epsilon) &\geq -\log(2) \Big/ \left[\frac{\pi}{2\tan^{-1}(1/2)} - 1\right] = -0.290 \\
\Rightarrow & \quad \boxed{\epsilon \geq e^{-0.29} = 0.748}
\end{aligned}$$

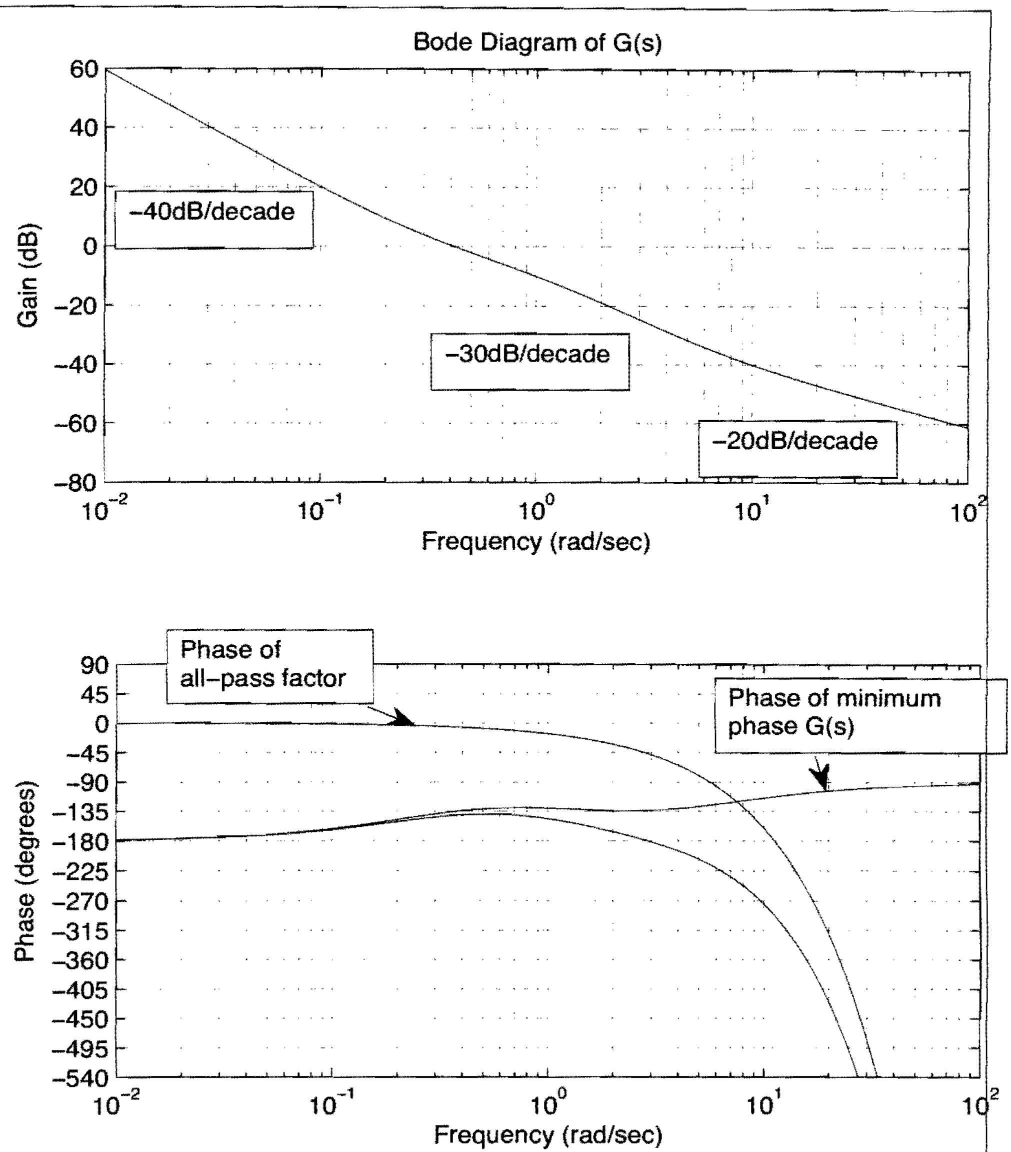
3. **Solution:**

(a) i. The number of poles at $s = 0$ is 2 since the Bode magnitude plot has a slope of -40 dB/decade at low frequency with the phase $\rightarrow -180°$ as $\omega \rightarrow 0$.

ii. The Bode gain/phase relationship for a stable minimum phase transfer function is given in the data sheet as,

$$\begin{aligned}
\angle G(j\omega_0) &\approx \frac{\pi}{2}\left.\frac{d\log|G(j\omega_0 e^{\nu})|}{d\nu}\right|_{\nu=0} \\
&= \begin{cases} -\pi/2 & \text{when slope} \quad -20\text{dB/decade} \\ -\pi & \text{when slope} \quad -40\text{dB/decade} \\ -3\pi/4 & \text{when slope} \quad -30\text{dB/decade} \end{cases}
\end{aligned}$$

which have been sketched on the Bode diagram.

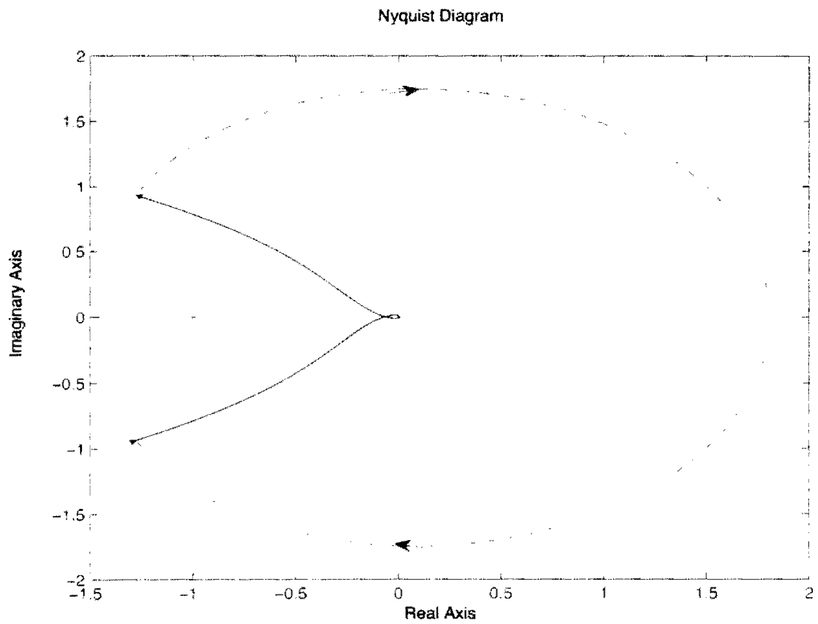


iii. The all-pass term looks like $e^{-j\omega T}$, with phase $= -\omega T$, and at $\omega = 20$ rad/s we have $\omega T \approx (425 - 105) * \pi/180 \Rightarrow T \approx 0.28$s.

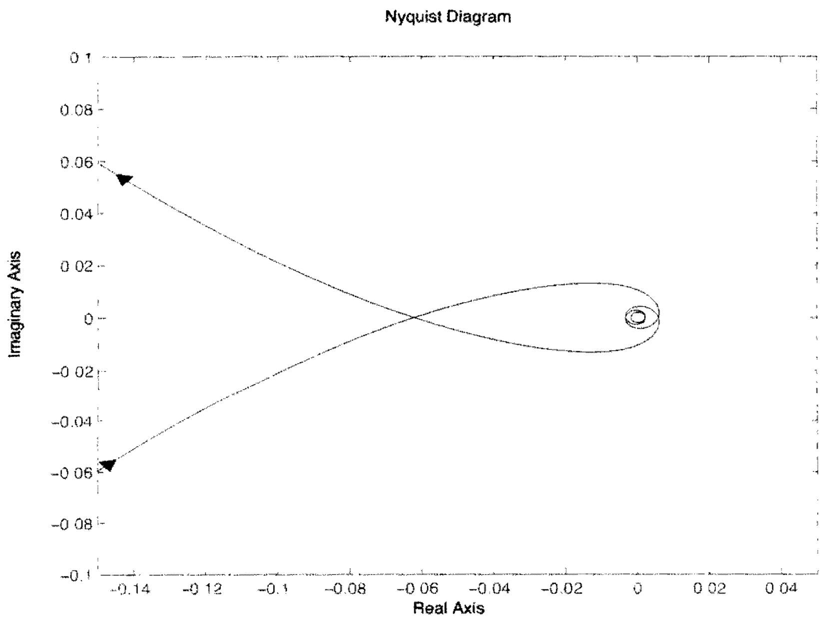
iv. It is difficult to obtain a crossover frequency for the closed loop system when the plant has a phase less than -180°. Hence a crossover frequency greater than 3 rad/s would be difficult.

v. The Nyquist diagram will be as below. Note that the large dashed part of a circle corresponds to the Nyquist D contour when it circles to the right of the two poles at $s = 0$. In this region we have $s = Re^{j\theta}$ where θ varies from $-\pi/2$ to $+\pi/2$ and R is small, and $G(s) \approx A/s^2$ with $A > 0$, so that $G \approx (A/R^2)e^{-2j\theta}$, so that the contour is as shown (rather than going the other way around). There are hence zero encirclements of the -1 point and no poles in the RHP and hence we have closed-loop stability if $k > 0$ and k is sufficiently small. To determine the maximum value of k we can examine the enlarged Nyquist plot below and deduce that closed-loop stability if: $\boxed{k > 0 \text{ and } k < 1/0.062 = 16}$

[aside : note if we had chosen to avoid the poles at $s = 0$ by going into the LHP then the Nyquist contour would have gone the other way around giving two counter-clockwise encirclements of the -1 point, and this will equal the number of RHP open-loop poles and hence again imply stability for the same range of k as before.]



The area near the origin has been magnified in the following plot:



(b) i. We are given the specification:

A: $|L(j\omega)| \geq 10$ for $\omega \leq 0.1$ rad s^{-1};

B: $|L(j\omega)| \leq 0.01$ for $\omega \geq 10$ rad s^{-1};

C: Phase margin of at least 45°.

Conditions A and B are just satisfied by unity gain constant controller $K = 1$. A phase advance compensator has $|K(0)| < |K(j\infty)|$ and $|K(0.1j)| < |K(10j)|$ so either A or B must be violated with a phase advance compensator.

The phase of a phase lag compensator is negative for all frequency, but the phase of the open loop plant is always less than -135° for all frequency and hence condition C cannot be satisfied with a phase lag compensator (since it would require the phase to be greater than -135° and the unity gain crossover frequency.

ii. A unity gain compensator almost satisfies C with a phase margin of $\approx 40°$. Taking the hint we will first design a phase lead compensator to increase the phase margin (but violate either A or B). Consider,

$$K_a(s) = \frac{s/0.5+1}{s+1}$$

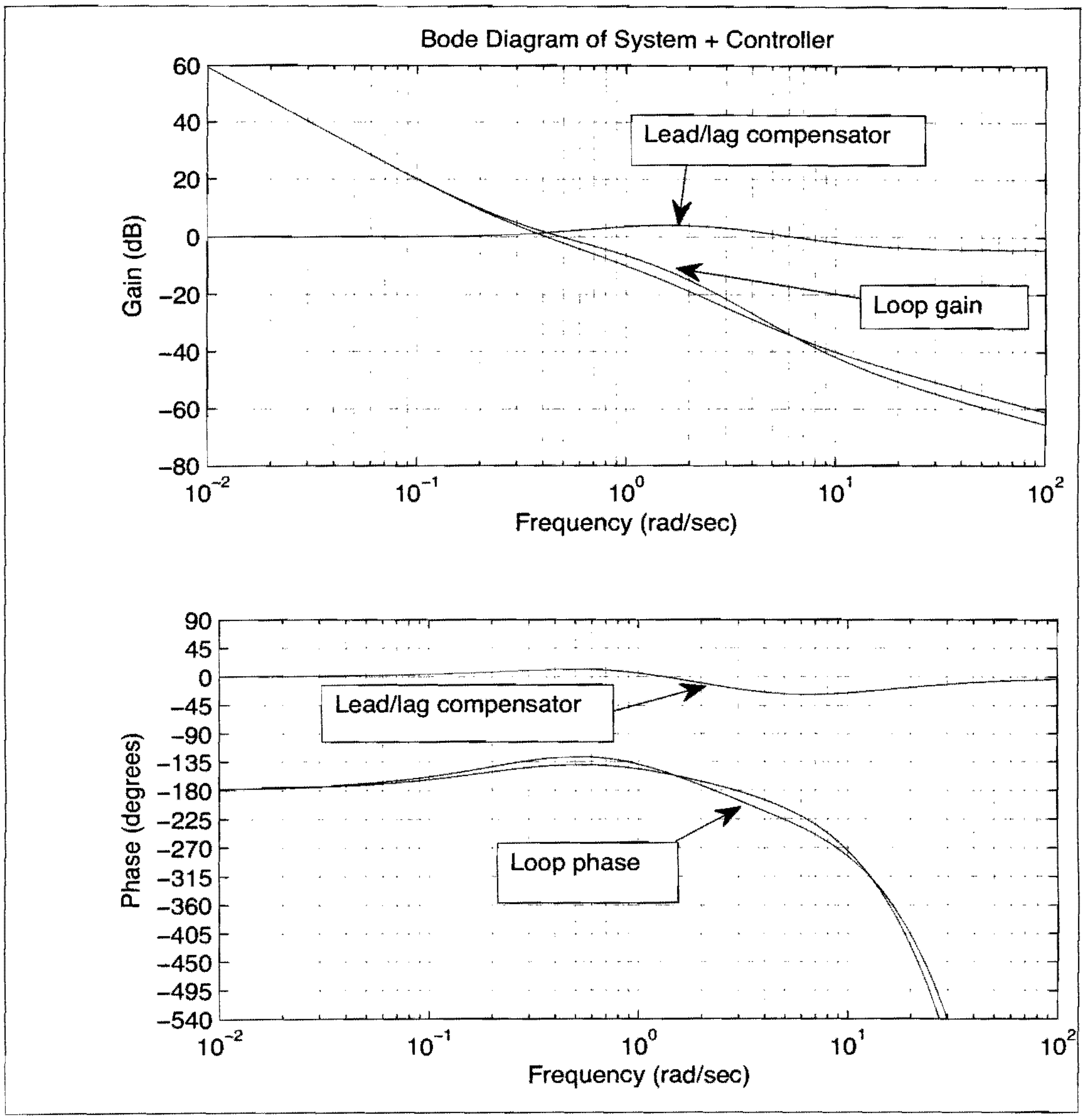
which will have little effect on the low frequency response but will increase the high frequency gain by a factor of 2 and add phase advance at $\omega = \omega_1 = \sqrt{0.5} = 0.707$ rad/s of $2\tan^{-1}(1/\sqrt{0.5}) - 90° = 19.5°$.

The high frequency gain is now too high and can be reduced by a phase lag compensator. Consider,

$$K_l(s) = \frac{0.1s+1}{s/3+1}$$

which will also have little effect on the low frequency gain but will reduce the gain at $\omega = 10$ rad/s to $|K_l(10j)| = \left|\frac{j+1}{10j/3+1}\right| = 0.4064$.

Conditions A and B are therefore met. Confirming condition C is not as easy since the crossover frequency will have to be determined. The accurate Bode diagrams for this arrangement are given below and it can be seen that the condition is met with the crossover frequency around 0.5 rad/s. *[The calculations in this solution have all been done accurately but clearly the graphical sketches asked for in the question will be much "broader brush" for which full credit was awarded.]*



Bode Diagram of System + Controller
Lead/lag compensator
Loop gain
Gain (dB)
Frequency (rad/sec)
Lead/lag compensator
Loop phase
Phase (degrees)
Frequency (rad/sec)

KG 2012