ENGINEERING TRIPOS PART IIB 2012
4F2 ROBUST AND NONLINEAR SYSTEMS AND CONTROL

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1. (a) The describing function is defined as $N(E) = (U_1 + jV_1)/E$, where U_1 and V_1 are the in-phase and quadrature first Fourier coefficients of $f(e)$, when $e(\theta) = E\sin(\theta)$. Since $f(e)$ is an odd function in this case, $V_1 = 0$. Assuming that $E > a$, U_1 is given by (the standard formula):

$$
\begin{aligned}
U_1 &= \frac{1}{\pi}\int_0^{2\pi} f(E\sin\theta)\sin\theta d\theta &(1)\\
&= \frac{2}{\pi}\int_0^{\pi} f(E\sin\theta)\sin\theta d\theta \quad \text{(since } f(e) \text{ is odd)} &(2)\\
&= \frac{4}{\pi}\int_0^{\pi/2} f(E\sin\theta)\sin\theta d\theta &(3)\\
&= \frac{4}{\pi}\int_{\sin^{-1}(a/E)}^{\pi/2} f(E\sin\theta)\sin\theta d\theta &(4)
\end{aligned}
$$

since $f(e) = 0$ for $0 \le e \le a$. Therefore

$$
\begin{aligned}
U_1 &= \frac{4}{\pi}\int_{\sin^{-1}(a/E)}^{\pi/2} (2E\sin\theta - a)\sin\theta d\theta &(5)\\
&= \frac{8E}{\pi}\int_{\sin^{-1}(a/E)}^{\pi/2} \sin^2\theta - \frac{4a}{\pi}\int_{\sin^{-1}(a/E)}^{\pi/2} \sin\theta d\theta &(6)\\
&= \frac{4E}{\pi}\int_{\sin^{-1}(a/E)}^{\pi/2} (1-\cos 2\theta)d\theta + \frac{4a}{\pi}\left[\cos\theta\right]_{\sin^{-1}(a/E)}^{\pi/2} &(7)\\
&= \frac{4E}{\pi}\left[\theta\right]_{\sin^{-1}(a/E)}^{\pi/2} - \frac{2E}{\pi}\left[\sin 2\theta\right]_{\sin^{-1}(a/E)}^{\pi/2} + \frac{4a}{\pi}\left[\cos\theta\right]_{\sin^{-1}(a/E)}^{\pi/2} &(8)\\
&= \left[2E - \frac{4E}{\pi}\sin^{-1}\left(\frac{a}{E}\right)\right] - \frac{2E}{\pi}\left[0 - \sin\left(2\sin^{-1}\left(\frac{a}{E}\right)\right)\right] + \frac{4a}{\pi}\left[0 - \cos\left(\sin^{-1}\left(\frac{a}{E}\right)\right)\right] &(9)
\end{aligned}
$$

Now from basic trigonometry we have that $\cos(\sin^{-1}x) = \sqrt{1-x^2}$ and $\sin(2\sin^{-1}x) = 2x\sqrt{1-x^2}$, so we have:

$$
\begin{aligned}
U_1 &= 2E - \frac{4E}{\pi}\sin^{-1}\left(\frac{a}{E}\right) + \frac{2E}{\pi}\left(\frac{2a}{E}\right)\sqrt{1-\left(\frac{a}{E}\right)^2} - \frac{4a}{\pi}\sqrt{1-\left(\frac{a}{E}\right)^2} &(10)\\
&= 2E - \frac{4E}{\pi}\sin^{-1}\left(\frac{a}{E}\right) &(11)
\end{aligned}
$$

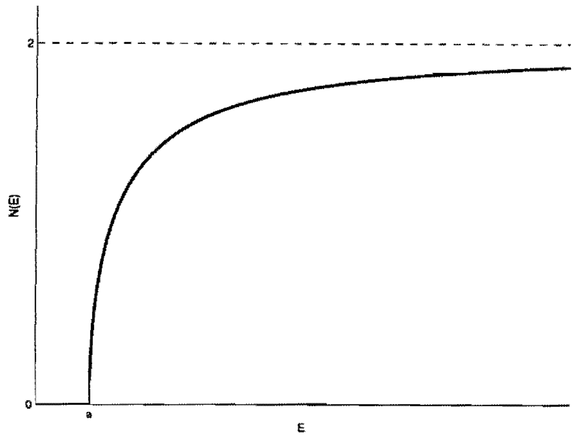

Figure 1: Describing function $N(E)$

Hence

$$N(E) = \frac{U_1}{E} = 2 - \frac{4}{\pi}\sin^{-1}\left(\frac{a}{E}\right) \tag{12}$$

If $E \le a$, then clearly $N(E) = 0$, since $f(E\sin\theta) = 0$.

(b) When $E > a$ but approaches a, we can see that $N(E)$ approaches 0, since $\sin^{-1}(a/E)$ approaches $\pi/2$. Thus $N(E)$ is continuous at $E = a$, and there is no jump in the graph. When $E \to \infty$ we have $\sin^{-1}(a/E) \to 0$, so $N(E) \to 2$. (This can also be seen by noticing that for very large input signals, the nonlinearity looks approximately like a straight line of slope 2.)
Does the graph have any turning points? Consider its slope. Since

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}} \tag{13}$$

we have (for $E > a$)

$$\frac{dN(E)}{dE} = -\frac{4}{\pi}\frac{(-a/E^2)}{\sqrt{1-\left(\frac{a}{E}\right)^2}} > 0 \quad \text{for all finite } E. \tag{14}$$

Thus we have $dN(E)/dE = \infty$ at $E = a$, and the slope reduces as E increases, but never becomes 0. The graph is therefore as shown in Fig.1.

(c) The describing function method predicts that a limit cycle will exist if the graph of $-1/N(E)$ intersects the Nyquist locus of $G(j\omega)$ on the Argand diagram. Since $N(E)$ is real, and increases monotonically from 0 to 2, the graph of $-1/N(E)$ lies on the real axis to the left of $-1/2$. So the question is where does the Nyquist locus of $G(j\omega)$ intersect the negative real axis?

$$\arg G(j\omega) = \arg\left(\frac{2}{(j\omega+1)^3}\right) = -3\arg(j\omega+1) = -3\tan^{-1}(\omega) \tag{15}$$

So the intersection with the negative real axis occurs at frequency ω_0, where

$$-3\tan^{-1}(\omega_0) = -\pi, \quad \text{hence} \quad \omega_0 = \tan\frac{\pi}{3} = \sqrt{3} \tag{16}$$

This gives

$$|G(j\omega_0)| = \frac{2}{|j\sqrt{3}+1|^3} = \frac{2}{2^3} = \frac{1}{4} \tag{17}$$

Since $1/4 < 1/2$, no intersection takes place between $-1/N(E)$ and $G(j\omega)$. Thus the describing function method predicts that no limit cycle will exist.

(d) The describing function method assumes that the linear system in the feedback loop has 'low-pass' characterisitcs, namely that if a limit cycle exists, then the second and higher harmonics propagating around the loop are negligible. The linear system in this example satisfies this assumption, since $|G(j\omega)|$ is monotonically decreasing. In particular, the gain falls approximately proportionally to ω^3 once the frequency is above the 'corner' (-3dB) frequency, which in this case is 1 rad/sec. The frequency on which the prediction is based is $\omega_0 = \sqrt{3} = 1.732$, which is a little above the corner frequency. So the neglected 2nd harmonic of this frequency would be attenuated approximately 8 times as much as the fundamental, the 3rd harmonic 27 times as much, etc. It therefore seems likely that the describing function prediction is reliable in this case.

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2. (a) Without loss of generality, assume that the equilibrium is at $x = 0$. Let $x(t; x_0)$ denote the solution of $\dot{x} = f(x)$ at time $t > 0$, if $x(0) = x_0$. The equilibrium is *stable* if, for any $\epsilon > 0$, it is possible to find a $\delta > 0$, such that $||x(t; x_0)|| < \epsilon$ for any x_0 such that $||x_0|| < \delta$.

(b) The linearisation of $\dot{x} = f(x)$ about the equilibrium x_e is given by $\dot{x} = Ax$, where

$$A = \left.\frac{\partial f}{\partial x}\right|_{x=x_e} \tag{18}$$

The stability of the equilibrium x_e can be investigated by checking the stability of the linearised system, namely by checking the eigenvalues of A. If the linearised system is asymptotically stable then the equilibrium x_e is stable. If the linearised system is unstable then the equilibrium x_e is unstable. If the linearised system is marginally stable, then the stability of x_e is not determined by this method. In terms of eigenvalues this means that

i. If all have negative real parts then the equilibrium x_e is stable.

ii. If any has a positive real part, or if there are any repeated eigenvalues on the imaginary axis, then the equilibrium x_e is unstable.

iii. Otherwise the stability of the equilibrium is not determined.

(c) i. At an equilibrium (x_{1e}, x_{2e}) we have $\dot{x}_1 = 0$ and $\dot{x}_2 = 0$. Therefore we have

$$0 = u(r_1 - x_{1e}) - kx_{1e}^2 \tag{19}$$

$$0 = u(r_2 - x_{2e}) + kx_{1e}^2 \tag{20}$$

From (19) we have

$$x_{1e} = \frac{1}{2k}\left(-u \pm \sqrt{u^2 + 4kur_1}\right) \tag{21}$$

Since $x_{1e} > 0$, the positive square root must be taken, so we have

$$x_{1e} = \frac{1}{2k}\left(-u + \sqrt{u^2 + 4kur_1}\right) \tag{22}$$

Noting from (19) that $kx_{1e}^2 = u(r_1 - x_{1e})$ and substituting this into (20) gives

$$u(r_1 - x_{1e}) = u(x_{2e} - r_2) \tag{23}$$

hence

$$x_{2e} = r_1 + r_2 - x_{1e} \tag{24}$$

(which can also be obtained by adding (19) and (20) together). It is possible that this value of x_{2e} turns out to be negative, in which case no equilibrium exists for $x_1 > 0$ and $x_2 > 0$. But if it is positive, then the equilibrium is unique, with the given values of x_{1e} and x_{2e}.

ii. The verification can be done either by substituting the given values of u, r_1, r_2, k into (22) and (24) or by checking that $(x_{1e}, x_{2e}) = (1, 2)$ satisfies (19) and (20).

iii. To check the stability of the equilibrium $(x_{1e}, x_{2e}) = (1, 2)$ we will linearise the nonlinear equations at this point. We have

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}_{(1,2)} \tag{25}$$

$$= \begin{bmatrix} -u - 2kx_1 & 0 \\ 2kx_1 & -u \end{bmatrix}_{(1,2)} \tag{26}$$

$$= \begin{bmatrix} -3 & 0 \\ 2 & -1 \end{bmatrix} \tag{27}$$

Since this is a triangular matrix the eigenvalues are just the diagonal elements, namely -3 and -1. So the linearised system is asymptotically stable, and hence the equilibrium $(x_{1e}, x_{2e}) = (1, 2)$ is stable.

Q1 Overall a straightforward question, well answered by most candidates. In part (b) most students failed to see the infinite slope at $E=a$. Part (d) was not well answered as the students could not explain rigorously the reason why the prediction is likely to be accurate.

Q2 A straightforward question, well answered by most candidates, which resulted in the highest mean of all three questions. Although very standard, the definition of stability was somewhat confusing to many students in part (a).

3. (a) Since the magnitude of the zero is smaller than the magnitude of the pole, the H_∞ norm is achieved as $\omega \to \infty$. Hence,

$$\left\| \frac{s+5}{s+7} \right\|_\infty = 1$$

(b) Firstly note that

$$\frac{1}{1+G} = \frac{1}{1+\frac{3}{s+4}} = \frac{s+4}{s+7}$$

is in H_∞ and

$$\frac{G}{1+G} = \frac{3}{s+7}$$

which is also in H_∞. Hence,

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{1+G} \begin{bmatrix} 1 & G \end{bmatrix}$$

is in H_∞. Then

$$\begin{aligned}
\bar{\sigma}^2 \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{1+G} \begin{bmatrix} 1 & G \end{bmatrix} \right\} &= \lambda_{\max} \left\{ \begin{bmatrix} 1 \\ G^* \end{bmatrix} \frac{1}{1+G^*} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{1+G} \begin{bmatrix} 1 & G \end{bmatrix} \right\} \\
&= 2\lambda_{\max} \left\{ \frac{1}{|1+G|^2} \begin{bmatrix} 1 \\ G^* \end{bmatrix} \begin{bmatrix} 1 & G \end{bmatrix} \right\} \\
&= 2\lambda_{\max} \left\{ \frac{1}{|1+G|^2} \begin{bmatrix} 1 & G \end{bmatrix} \begin{bmatrix} 1 \\ G^* \end{bmatrix} \right\} \\
&= 2 \frac{1+|G|^2}{|1+G|^2}
\end{aligned}$$

where we used the fact that $\lambda_i(AB) = \lambda_i(BA)$ for $\lambda_i \neq 0$. Hence,

$$\begin{aligned}
\left\| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \frac{1}{1+G} \begin{bmatrix} 1 & G \end{bmatrix} \right\|_\infty^2 &= 2 \sup_\omega \frac{1 + \frac{9}{\omega^2+16}}{\left| 1 + \frac{3}{j\omega+4} \right|^2} \\
&= 2 \sup_\omega \frac{\omega^2+25}{\omega^2+49} \\
&= 2
\end{aligned}$$

from part (a).

(c) $G = \frac{3}{s+4} = N/M$. Normalisation requires that $|N|^2 + |M|^2 = 1$ or

$$\frac{1}{|N|^2} = 1 + \frac{|M|^2}{|N|^2} = 1 + \frac{1}{|G|^2}.$$

Thus,

$$\frac{1}{|N|^2} = 1 + \frac{\omega^2+16}{9} \quad \Rightarrow \quad N^2 = \frac{9}{\omega^2+25}.$$

Take

$$N(s) = \frac{3}{s+5} \;;\;\; M(s) = \frac{s+4}{s+5}$$

(d) $G_\Delta = (N + \Delta_N)/(M + \Delta_M)$, $\left\| [\Delta_N \quad \Delta_M] \right\|_\infty < \epsilon$. This set is stabilised by the controller $k = -1$ if

$$\left\| \begin{bmatrix} -1 \\ 1 \end{bmatrix} (1+G)^{-1} [1 \quad G] \right\|_\infty \leq 1/\epsilon$$

i.e. $\epsilon \leq 1/\sqrt{2}$.

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4. (a) Using the hint, $y = d + P_1 u$ and also $u = v + K(r + y - P_2 u)$. Replacing y from the first equation in the second equation gives $u = v + K(r + d + P_1 u - P_2 u)$. Solving for u yields

$$u = [I - K(P_1 - P_2)]^{-1} (Kr + v + Kd)$$

Replacing u in y gives

$$y = P_1 [I - K(P_1 - P_2)]^{-1} (Kr + v + Kd) + d$$

This shows the first two transfer functions. The transfer function from d to y so far is

$$P_1 [I - K(P_1 - P_2)]^{-1} K + I$$

Multiplying on the right by $(I - (P_1 - P_2)K)$

$$\begin{aligned} & P_1 [I - K(P_1 - P_2)]^{-1} K(I - (P_1 - P_2)K) + I - (P_1 - P_2)K \\ &= P_1 [I - K(P_1 - P_2)]^{-1} (K - K(P_1 - P_2)K) + I - (P_1 - P_2)K \\ &= P_1 [I - K(P_1 - P_2)]^{-1} (I - K(P_1 - P_2))K + I - (P_1 - P_2)K \\ &= P_1 K + I - P_1 K - P_2 K \\ &= I - P_2 K \end{aligned}$$

(b) The transfer function from r to y is $P_1 K$. Hence, since K is unstable, the closed-loop system is unstable.

(c) The transfer function from v to y is $\frac{1}{s-1}$, which is clearly unstable. Hence, there is no K that stabilises the system.

(d) We have that

$$\sigma_{max}[(P_1(j\omega) - P_2(j\omega))K(j\omega)] \leq \sigma_{max}[P_1(j\omega) - P_2(j\omega)] \, \sigma_{max}[K(j\omega)] \leq c < 1, \quad \forall \omega$$

Hence

$$\sup_\omega \sigma_{max}[(P_1(j\omega) - P_2(j\omega))K(j\omega)] < 1$$

Since P_1, P_2, K are all stable, this is sufficient to ensure stability of the closed loop system.

(e) Take $P_1(s) = \frac{2}{s+1}$, $P_2(s) = \frac{1}{s+1}$ and $K(s) = -10$. Then the closed-loop system is stable (closed-loop pole at $s = -11$). However, condition (1) is not satisfied.

Q3.

Overall a straightforward question, well answered by most candidates. In part (b) many students forgot about the transpose conjugate in calculating the maximum singular value. Others, calculated the eigenvalues of matrix transfer function directly. In part (d) some students did not notice that the required stability condition was already given in part (b).

Q4

This was the question that was the least attempted. It clearly suffered from lack of time as most students did not even tried parts (c), (d) and (e). Hence, the low average.