ENGINEERING TRIPOS PART IIB 2012
4F6 SIGNAL DETECTION AND ESTIMATION **Prof W J Fitzgerald**

(Solutions)

Q1) Bookwork definitions – see lecture notes.

Neyman-Fisher factorization theorem:

For an unbiased efficient estimator,

$$\frac{\partial}{\partial\theta} \ln p(x|\theta) = k(\theta)\,(\hat{\theta}(x) - \theta)$$

$$\therefore \ln p(x|\theta) = \int k(\theta')\left(\hat{\theta}(x) - \theta'\right)d\theta' + \ln h(x)$$

$$\therefore p(x|\theta) = g(T(x),\theta)\; h(x)$$

30%

where $T(x)$ is a sufficient statistic for θ.

For the scalar exponential family, the likelihood of the data $\underline{x}$ is given by.

$$p(\underline{x}|\theta) = \prod_{n=0}^{N-1} \exp\left(A(\theta)B(x) + C(x) + D(\theta)\right)$$

$$= \exp\left(A(\theta)\sum_n B(x_n) + \sum_n C(x_n) + N D(\theta)\right)$$

$$= \exp\left(A(\theta)\sum_n B(x_n) + N D(\theta)\right) \cdot \exp\left(\sum_n C(x_n)\right)$$

$$= g(T(\underline{x}),\theta)\; h(x).$$

By NF factorization theorem, $T(x) = \sum_n B(x_n)$ 35%

Q1 cont.)

For the Gaussian case, we can write

$$p(x|\mu) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(x-\mu)^2\right)$$

taking $\sigma = 1$ for simplicity

$$\therefore p(x|\mu) = \exp\Big(\underbrace{x\mu}_{A(\mu)B(x)} \underbrace{-\tfrac{1}{2}x^2}_{C(x)} + \underbrace{\left(-\tfrac{1}{2}\mu^2 + \ln \tfrac{1}{\sqrt{2\pi}}\right)}_{D(\mu)}\Big).$$

$$\therefore \quad T(\underline{x}) = \sum_{n=0}^{N-1} x_n$$

For the exponential case

$$p(x|\lambda) = \lambda e^{-\lambda x}$$

$$= \exp\Big(\underbrace{-\lambda x}_{A(\lambda)B(x)} + \underbrace{0}_{C(x)} + \underbrace{\ln \lambda}_{D(\lambda)}\Big)$$

$$\therefore T(x) = \sum_{n=0}^{N-1} x_n$$

35%.

Q2) Bookwork definitions – see lecture notes.

a) This distribution belongs to the Rayleigh family and

$$p(\underline{x}|\theta) = \prod_{i=1}^{N} p(x_i|\theta)$$

$$= \frac{1}{\theta^N} \left(\prod_{i=1}^{N} x_i \right) e^{-\frac{1}{2\theta} \sum_{i=1}^{N} x_i^2}$$

The score function is given by:

$$S(\theta, x_n) = \frac{\partial}{\partial \theta} \ln p(x_n|\theta)$$

$$= -\frac{N}{\theta} + \frac{1}{2\theta^2} \sum_{i=1}^{N} x_i^2$$

Therefore the expected value of S is zero and the Cramer-Rao bound applies

$$\therefore \frac{\partial}{\partial \theta} \ln p(x_n|\theta) = \frac{n}{\theta^2} \left(\frac{1}{2N} \sum_{i=1}^{N} x_i^2 - \theta \right)$$

$$\therefore \hat{\theta} = \frac{1}{2N} \sum^{N} x_i^2$$ ML estimator

25%

Q2 cont.)

b) Since the expected value of $s = 0$, the estimator is unbiased

15%

c) The Fisher information is obtained as the pre-factor in the score function

$$\therefore \quad I(\theta) = \frac{N}{\theta^2}$$

and $\therefore$ the CRLB is

$$\mathrm{Var}\left(\hat{\theta}\right) \geq \frac{\theta^2}{N}.$$

15%.

d) From $p(x_n|\theta) = \frac{1}{\theta^N} \left(\prod_{i=1}^{N} x_i \right) e^{\left(-\frac{1}{2\theta} \sum_{i=1}^{N} x_i^2 \right)}$

and using the Neyman-Fisher factorization theorem, a sufficient statistic for θ is

$$\sum_{i=1}^{N} x_i^2$$

10%

Q3) Parts a) and b) are bookwork
– see lecture notes.

c) i) The general linear model may be written:

$$\underline{d} = \underline{G}\,\underline{\theta} + \underline{w}$$

Where the model parameters are contained in the vector $\underline{\theta}$ and $\underline{d}$ is the observed data vector, $\underline{w}$ is the noise vector and G is a matrix.

For the signal $s(n) = A + Bn$, the observed data is $\underline{d} = \underline{s} + \underline{w}$

$$\therefore \underline{d} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ \vdots & \vdots \\ 1 & N-1 \end{bmatrix} \begin{bmatrix} A \\ B \end{bmatrix} + \underline{w}$$

$$\therefore \underline{d} = \underline{G}\,\underline{\theta} + \underline{w}$$

30%.

Q3

c) ii) The two likelihoods are:

$$p(\underline{d}\,|H_0) = \frac{1}{(2\pi)^{N/2}\,|C|^{1/2}}\, e^{-\frac{1}{2} d^T C^{-1} d}$$

$$p(\underline{d}\,|H_1) = \frac{1}{(2\pi)^{N/2}\,|C|^{1/2}}\, e^{-\frac{1}{2}(\underline{d}-\underline{s})^T C^{-1} (\underline{d}-\underline{s})}$$

The N-P detector is given by

$$L(\underline{d}) = \frac{p(\underline{d}\,|H_1)}{p(\underline{d}\,|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} \lambda$$

$$\therefore L(\underline{d}) = e^{-\frac{1}{2}(-2d^T C^{-1} s + s^T C^{-1} s)}.$$

$$\therefore \quad d^T C^{-1} S \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2} S^T C^{-1} S + \ln \lambda.$$

Therefore the detector can be written

$$\frac{1}{\sigma^2} \sum^{T} d(n)\,(A + Bn) \underset{H_0}{\overset{H_1}{\gtrless}} \lambda'$$

30%.

Q4)

First two parts are bookwork - see notes.

To derive the MAP detector, we need to consider the likelihood ratios.

$$p(\underline{y}|H_0) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp -\frac{1}{2\sigma^2}(\underline{y}-\underline{s}_0)^T(\underline{y}-\underline{s})$$

$$p(\underline{y}|H_1) = \frac{1}{(2\pi\sigma^2)^{N/2}} \exp -\frac{1}{2\sigma^2}(\underline{y}-\underline{s}_1)^T(\underline{y}-s_1).$$

$$\therefore \; L(\underline{y}) = \frac{p(\underline{y}|H_1)}{p(\underline{y}|H_0)} \underset{H_0}{\overset{H_1}{\gtrless}} k.$$

$$\therefore \; \underline{y}^T(\underline{s}_1-\underline{s}_0) \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2}(\underline{s}_1^T\underline{s}_1-\underline{s}_0^T\underline{s}_0) + \sigma^2 \ln k.$$

For the case of coloured noise with covariance matrix C we have

$$\underline{y}^T C^{-1} s_1 \gtrless \frac{1}{2} s_1^T C^{-1} s_1 + \ln k.$$

30%

taking the case $s_0 = 0$ for simplicity

Q4)
cont) The noise covariance matrix is symmetric positive semi-definite and may be diagonalised with an eigenvector ~~transformation~~

$$C = U \Lambda U^{-1}$$

U – square, Λ – diagonal

Since C is real symmetric,

$$C = U \Lambda U^T$$

$$C^{-1} = U \Lambda^{-1} U^T$$

$$= D D^T$$

where $D = U \Lambda^{-1/2}$

$$\therefore y^T D D^T s_1 \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2} s_1^T D D^T s_1 + \ln k$$

Define $y' = D^T y$, $s_1' = D^T s_1$

$$\therefore y'^T s_1' \underset{H_0}{\overset{H_1}{\gtrless}} \frac{1}{2} s_1'^T s_1' + \ln k$$

20%

D is a pre-whitening filter