ENGINEERING TRIPOS PART IIB 2012 DR SS SINGH

Module 4F7

DIGITAL FILTERS AND SPECTRUM ESTIMATION

1 Solution to Question 1

a) The M-tap LMS algorithm for linear prediction is

$$\mathbf{h}(n+1) = \mathbf{h}(n) + \mu(u(n) - \mathbf{h}(n)^{\mathrm{T}}\mathbf{u}(n-1))\mathbf{u}(n-1)$$

where $\mathbf{u}(n-1) = [u(n-1), u(n-2), \ldots, u(n-M)]^{\mathrm{T}}$.

b-i) Take the expectation of both sides to get

$$\begin{aligned}
E\{\mathbf{h}(n+L)\} &= E\{\mathbf{h}(n)\} + \mu\frac{1}{L}\sum_{l=0}^{L-1}\left(E\{d(n+l)\mathbf{u}(n+l)\} - E\{\mathbf{u}(n+l)\mathbf{u}(n+l)^{\mathrm{T}}\mathbf{h}(n)\}\right) \\
&= E\{\mathbf{h}(n)\} + \mu\frac{1}{L}\sum_{l=0}^{L-1}\left(\mathbf{p} - E\{\mathbf{u}(n+l)\mathbf{u}(n+l)^{\mathrm{T}}\mathbf{h}(n)\}\right) \\
&\approx E\{\mathbf{h}(n)\} + \mu\frac{1}{L}\sum_{l=0}^{L-1}\left(\mathbf{p} - E\{\mathbf{u}(n+l)\mathbf{u}(n+l)^{\mathrm{T}}\}E\{\mathbf{h}(n)\}\right) \\
&= E\{\mathbf{h}(n)\} + \mu\frac{1}{L}\sum_{l=0}^{L-1}\left(\mathbf{p} - \mathbf{R}E\{\mathbf{h}(n)\}\right) \\
&= (I - \mu\mathbf{R})\,E\{\mathbf{h}(n)\} + \mu\mathbf{p}
\end{aligned}$$

where $E\{d(n+l)\mathbf{u}(n+l)\} = \mathbf{p}$ and $E\{\mathbf{u}(n+l)\mathbf{u}(n+l)^{\mathrm{T}}\} = \mathbf{R}$.

Convergence in mean provided $0 < \mu < 2/\lambda_{\max}(\mathbf{R})$

We see that the block performs an average of the update terms which should be closer to their mean value than when $L = 1$ and thus oscillation about the converged value should be less. Also, the independence assumption is a more realistic approximation for the block LMS.

b-ii) It is clear that the block LMS has the same limit point for all L. Call this $\mathbf{h}$.

$$\begin{aligned}
\mathbf{h} &= (I - \mu\mathbf{R})\,\mathbf{h} + \mu\mathbf{p} \\
\mathbf{h} &= \mathbf{R}^{-1}\mathbf{p}
\end{aligned}$$

$$\begin{aligned}
\mathbf{R} &= E\{\mathbf{u}(n)\mathbf{u}(n)^{\mathrm{T}}\} \\
&= E\{(\mathbf{x}(n) + \mathbf{v}(n))(\mathbf{x}(n) + \mathbf{v}(n))^{\mathrm{T}}\} \\
&= E\{\mathbf{x}(n)\mathbf{x}(n)^{\mathrm{T}} + \mathbf{x}(n)\mathbf{v}(n)^{\mathrm{T}} + \mathbf{v}(n)\mathbf{x}(n)^{\mathrm{T}} + \mathbf{v}(n)\mathbf{v}(n)^{\mathrm{T}}\} \\
&= \mathbf{R}_x + \sigma_v^2\mathbf{I}
\end{aligned}$$

where $\mathbf{I}$ is the $M \times M$ identity matrix, $\mathbf{x}(n) = [x(n), x(n-1), \ldots, x(n-M+1)]^{\mathrm{T}}$, $\mathbf{v}(n) = [v(n), v(n-1), \ldots, v(n-M+1)]^{\mathrm{T}}$.

$$\lim_{n\to\infty} E\{\mathbf{h}(n)\} = (\mathbf{R}_x + \sigma_v^2)^{-1} E\{\mathbf{u}(n-1)u(n)\}$$

and

$$\begin{aligned}\mathbf{p} &= E\{\mathbf{u}(n-1)u(n)\} = E\{(\mathbf{x}(n-1) + \mathbf{v}(n-1))(x(n) + v(n))\} \\ &= E\{(\mathbf{x}(n-1)x(n)\}\end{aligned}$$

Examiner's comments: With the exception of parts (a) and (c-ii), the question was answered well. It was a disappointment to see that candidates were unable to formulate the prediction problem using LMS in part (a). A simple diagram would have helped. Also, many candidates failed to realise, as the question clearly states, that signal x(n) is not measured directly and hence cannot be used as the desired response.

2 Solution to Question 2

a) $E\left\{\left(\frac{1}{n}\sum_{i=1}^{n} y(i) - x\right)^2\right\} = E\left\{\left(\frac{1}{n}\sum_{i=1}^{n} v(i)\right)^2\right\} = \frac{\sigma_v^2}{n}$

b)

$$\begin{aligned}\widehat{x}(n) &= \frac{1}{n}y(n) + \frac{n-1}{n}\widehat{x}(n-1) \\ &= \frac{1}{n}\;(y(n) - \widehat{x}(n-1)) + \widehat{x}(n-1)\end{aligned}$$

c) Let $e(n) = \widehat{x}(n) - x$. Thus

$$\begin{aligned}\widehat{x}(n) - x &= K(n)\;(y(n) - x + x - \widehat{x}(n-1)) + \widehat{x}(n-1) - x \\ e(n) &= K(n)\;(v(n) - e(n-1)) + e(n-1)\end{aligned}$$

Square it and take the expectation to get:

$$\begin{aligned}e(n)^2 &= K(n)^2\,(v(n) - e(n-1))^2 + e(n-1)^2 \\ &+ 2K(n)\,(v(n) - e(n-1))\,e(n-1)\end{aligned}$$

$$\begin{aligned}E\left\{e(n)^2\right\} &= K(n)^2 E\left\{v(n)^2 + e(n-1)^2 - 2v(n)e(n-1)\right\} + E\left\{e(n-1)^2\right\} \\ &+ 2K(n)E\left\{(v(n) - e(n-1))\,e(n-1)\right\} \\ &= K(n)^2\left(\sigma_v^2 + E\left\{e(n-1)^2\right\}\right) + E\left\{e(n-1)^2\right\} \\ &- 2K(n)E\left\{e(n-1)^2\right\}\end{aligned}$$

The last line follows from the stated assumption on $\{v(n)\}$. Let $\sigma(n)^2 = E\left\{e(n)^2\right\}$.

d) Differentiating the right-hand side with respect to $K(n)$ and equating to 0 to solve for $K(n)$ yields:

$$K(n) = \frac{\sigma(n-1)^2}{\sigma_v^2 + \sigma(n-1)^2}.$$

Examiner's comments: A well answered question on the whole. It was noted that more than a few candidates were unable to answer part (b). Although this part was only worth 5%, it was very surprising for the examiner.

3 Solution to Question 3

a) The AR model is an all pole IIR filter driven by white noise. The transfer function is

$$H(z) = \frac{\sigma}{A(z)}$$

where:

$$A(z) = 1 + \sum_{i=1}^{P} a_i z^{-i}$$

Assume that the filter is stable, i.e. the poles (solutions of $A(z) = 0$) all lie within the unit circle. Otherwise the output wont be wide sense stationary. The power spectrum of the output is

$$S_X(e^{j\omega T}) = \frac{\sigma^2}{|A(e^{j\omega T})|^2}$$

The spectrum can be factored as

$$\frac{\sigma}{A(z)} = \frac{\sigma}{\prod_{i=1}^{P}(1 - z^{-1}d_i)}$$

The spectrum can be manipulated by choosing P, σ, $\{d_i\}_{i=1}^{P}$ subject to $|d_i| < 1$. The poles model well the peaks in the spectrum - sharper peaks implies poles closer to the unit circle

b) The autocorrelation function $R_{XX}[r]$ for the output x_n of the ARMA model is:

$$R_{XX}[r] = E[x_n x_{n+r}].$$

Note that $R_{XX}[r] = R_{XX}[-r]$.

Substituting the difference equation for x_{n+r} , $r \geq 0$, gives:

$$
\begin{aligned}
&R_{XX}[r] \\
&= E\left[x_n\left\{-\sum_{i=1}^{P} a_i\, x_{n+r-i} + \sigma\, w_{n+r}\right\}\right] \\
&= -\sum_{i=1}^{P} a_i\, E[x_n x_{n+r-i}] + \sigma\, E[x_n w_{n+r}] \\
&= -\sum_{i=1}^{P} a_i\, R_{XX}[r-i] + \sigma^2 \delta(r)
\end{aligned}
$$

where $\delta(0) = 1$, $\delta(r) = 0$ for $r \neq 0$.

c) In matrix form:

$$
\begin{bmatrix}
R_{XX}[0] & R_{XX}[-1] & \dots & R_{XX}[-P] \\
R_{XX}[1] & R_{XX}[0] & \dots & R_{XX}[1-P] \\
\vdots & \vdots & & \vdots \\
R_{XX}[P] & R_{XX}[P-1] & \dots & R_{XX}[0]
\end{bmatrix}
\times
\begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_P \end{bmatrix}
=
\begin{bmatrix} \sigma^2 \\ 0 \\ \vdots \\ 0 \end{bmatrix}
$$

Consider the matrix equation given by the second row onwards:

$$
\begin{bmatrix}
R_{XX}[0] & R_{XX}[-1] & \dots & R_{XX}[1-P] \\
R_{XX}[1] & R_{XX}[0] & \dots & R_{XX}[2-P] \\
\vdots & \vdots & & \vdots \\
R_{XX}[P-1] & R_{XX}[P-2] & \dots & R_{XX}[0]
\end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_P \end{bmatrix}
= -
\begin{bmatrix} R_{XX}[1] \\ R_{XX}[2] \\ \vdots \\ R_{XX}[P] \end{bmatrix}
$$

Solve for a_i. Then, use the first row of the original matrix equation to solve for σ:

$$
\sigma^2 = \begin{bmatrix} R_{XX}[0] & R_{XX}[-1] & \dots & R_{XX}[-P] \end{bmatrix}
\begin{bmatrix} 1 \\ a_1 \\ \vdots \\ a_P \end{bmatrix}
$$

d) The autocorrelation values of a particular signal are estimated to be

$$
\widehat{R}_{XX}[0] = 4.8, \qquad \widehat{R}_{XX}[1] = -1.2, \qquad \widehat{R}_{XX}[2] = 1.8
$$

This gives

$$\begin{bmatrix} \widehat{R}_{XX}[0] & \widehat{R}_{XX}[-1] \\ \widehat{R}_{XX}[1] & \widehat{R}_{XX}[0] \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} \widehat{R}_{XX}[1] \\ \widehat{R}_{XX}[2] \end{bmatrix},$$

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} 4.8 & -1.2 \\ -1.2 & 4.8 \end{bmatrix}^{-1} \begin{bmatrix} -1.2 \\ 1.8 \end{bmatrix} = \begin{bmatrix} 1/6 \\ -1/3 \end{bmatrix},$$

$$\sigma^2 = 4.8 - \frac{1.2}{6} - \frac{1.8}{3} = 4$$

e) $A(z) = 1 + \frac{1}{6}z^{-1} - \frac{1}{3}z^{-2} = (-\frac{1}{2}z^{-1} + 1)(\frac{2}{3}z^{-1} + 1)$

$$\frac{\sigma}{A(e^{j\omega})} = \frac{2}{(-\frac{1}{2}e^{-j\omega} + 1)(\frac{2}{3}e^{-j\omega} + 1)}$$

There will be two peaks in the spectrum at $\omega = 0$ and $\omega = \pi$

Examiner's comments: The most popular question. It was disappointing to see that most of the discussions for part (a) were poorly structured and incomplete while it should have been an easy mark earner. Sketching the power spectrum in part (e) was a stumbling block for many while all that was needed was simple reasoning to indicate the presence of two peaks and their locations.

4 Solution to Question 4

a)

$$\begin{aligned} E\{y_n y_{n+k}\} &= E\{(x_n + v_n)(x_{n+k} + v_{n+k})\} \\ &= E\{x_n x_{n+k}\} + E\{v_n v_{n+k}\} + \text{crossterms} \end{aligned}$$

Note that the cross terms have zero expectation. So

$$R_{YY}[k] = R_{XX}[k] + R_{VV}[k]$$

b)

$$A(z) = 1 + a_1 z^{-1} + a_2 z^{-2}, \qquad S_X(e^{j\omega}) = \sigma^2 |A(e^{j\omega})|^{-2}$$

$$B(z) = b_0 + b_1 z^{-1}, \qquad S_V(e^{j\omega}) = |B(e^{j\omega})|^2$$

$$S_Y(e^{j\omega}) = S_X(e^{j\omega}) + S_V(e^{j\omega})$$

c)

$$\begin{aligned} S_V(e^{j\omega}) &= |b_0 + b_1(\cos\omega - j\sin\omega)|^2 \\ &= b_0^2 + b_1^2\cos^2\omega + 2b_1b_0\cos\omega + b_1^2\sin^2\omega \\ &= b_0^2 + b_1^2 + 2b_1b_0\cos\omega \end{aligned}$$

We see that $b_1 b_0 = 1$, $b_0^2 + b_1^2 = 2$ or

$$(b_0 + b_1)^2 = 2 + 2b_1 b_0 = 4$$

$$\begin{aligned} b_0 + b_1 = 2, \quad &\text{or} \quad b_0 + b_1 = -2 \\ b_0^2 + 1 = 2b_0, \quad &\text{or} \quad b_0^2 + 1 = -2b_0 \\ b_0 = 1, b_1 = 1 \quad &\text{or} \quad b_0 = -1, b_1 = -1 \end{aligned}$$

Use $b_0 = b_1 = 1$, $v_n = e_n + e_{n-1}$. So

$$R_{VV}[0] = 2, \ R_{VV}[1] = 1, \ R_{VV}[2] = 0, \ldots$$

d) We can estimate $\widehat{R}_{XX}$ using the given $\widehat{R}_{YY}$ and calculated $\widehat{R}_{VV}$

$$\widehat{R}_{XX}[0] = 2.74, \ \widehat{R}_{XX}[1] = -0.46, \ \widehat{R}_{XX}[2] = 1.41,$$

Now use the Yule-Walker equations

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = - \begin{bmatrix} 2.74 & -0.46 \\ -0.46 & 2.74 \end{bmatrix}^{-1} \begin{bmatrix} -0.46 \\ 1.41 \end{bmatrix} \approx \begin{bmatrix} 1/12 \\ -1/2 \end{bmatrix},$$

$$\sigma^2 = 2.74 - \frac{0.46}{12} - \frac{1.41}{2} \approx 2$$

Examiner's comments: It was indeed pleasing to see that this entire question was answered well by most.