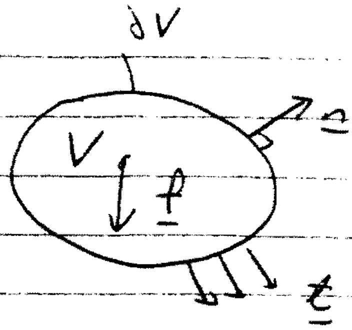
1 a)



$$\underline{t} = \underline{\underline{\sigma}} n \rightarrow t_i = \sigma_{ij} n_j$$

Equilibrium (balance of forces)

$$\int_{\partial V} \underline{t} \, ds + \int_V \underline{f} \, dV = \underline{0}$$

$$= \int_{\partial V} \underline{\underline{\sigma}} \, \underline{n} \, ds + \int_V \underline{f} \, dV = \underline{0}$$

$$= \int_{\partial V} \sigma_{ij} n_j \, ds + \int_V f_i \, dV = 0$$

Apply divergence theorem

$$\int_V \frac{d\sigma_{ij}}{dx_j} \cancel{dV} + f_i \, dV = 0$$

Localise,

$$\nabla \cdot \underline{\underline{\sigma}} + \underline{f} = 0 \quad (\sigma_{ij,j} + f_i = 0)$$

Insert expression for $\underline{\underline{\sigma}} = 2\nu \underbrace{\nabla^s \underline{u}}_{\varepsilon_{ij}} - p \underline{\underline{I}}_{\delta_{ij}}$

$$\rightarrow (2\nu \varepsilon_{ij} - p \delta_{ij})_{,j} + f_i = 0$$

1 b) $\nabla \cdot 2\nu \nabla^s \underline{u} = \nu (u_{i,j} + u_{j,i})_{,j}$

$= \nu (u_{i,jj} + \cancel{u_{j,ij}}^{0})$

~~since $u_{i,j}$~~ since $u_{i,ij} = \underbrace{(u_{j,j})}_{}{}_{,i}$

$= \nu \nabla \cdot \nabla u + \nu \nabla (\cancel{\nabla \cdot u})_0$

$\nabla \cdot \underline{u} = 0$

c) $I = J + \underbrace{\int_V \lambda \nabla \cdot \underline{u} \, dV}_{\text{incompressibility constraint}}$

- $DI(u,\lambda)[\underline{v}] = \dfrac{d}{d\varepsilon}\Big[\int_V \nu \nabla^s (\underline{u} + \varepsilon \underline{v}) : \nabla^s (\underline{u} + \varepsilon \underline{v}) + \lambda \nabla \cdot (\underline{u} + \varepsilon \underline{v}) \, dV \Big]_{\varepsilon}$

$= \int_V 2\nu \nabla^s (\underline{u} + \varepsilon \underline{v}) : \nabla^s \underline{v} + \lambda \nabla \cdot \underline{v} \, dV \Big|_{\varepsilon = 0}$

$= \int_V 2\nu \nabla^s \underline{u} : \nabla^s \underline{v} + \lambda \nabla \cdot \underline{v} \, dV$

Apply integration by parts,

$= \cancel{\#} \int_V \cancel{2} \nu \underline{A_{ij}} (v_{i,j} + v_{j,i}) + \lambda v_{i,i} \, dV$

$A_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$

$\cancel{= \int}$ Note $A_{ij} v_{j,i} = A_{ji} v_{j,i}$

$= A_{ij} v_{i,j}$

(symmetry)

1c) cont

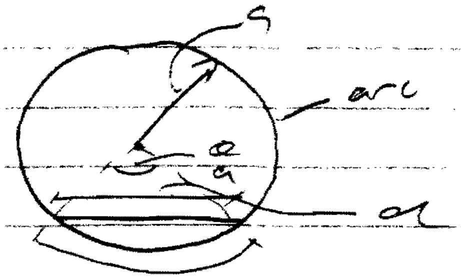
$$= -\int 2(\nu A_{ij})_{,j} v_i + \lambda_{,i} v_i \, dV$$
+ boundary term.

By ~~the~~ usual arguments,

$$\cancel{= 2\nu}$$
$$-2\nabla\cdot\nu\nabla u + \nabla\lambda = \underline{f}$$

$\longrightarrow$ pressure is a Lagrange multiplier

2c)



$l = \cancel{2\pi a}\ \pi a/2$

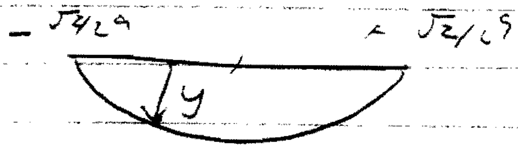
$\theta a = \pi a/2$

$\to \theta = \pi/2$

$\therefore\ d = 2a\sin(\pi/2) = \sqrt{2}a$

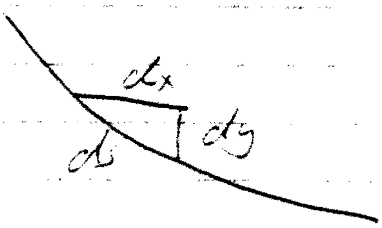
b) Potential energy

$\cancel{x}\ I = -\int_{-\pi a/4}^{\pi a/4} \rho g y\, ds$



(gravity)

Constraint $\int_{-\pi a/4}^{\pi a/4} dx = \sqrt{2}a$



$\frac{dy}{ds} ds = dy = y' ds$

$ds^2 = dx^2 + y'^2 ds^2$

$\to dx^2 = (1-y'^2) ds^2$

$\int_{-\pi a/4}^{\pi a/4} (1-y'^2)^{\frac{1}{2}} ds = \sqrt{2}a$

2c Need to minimise I subject to constraint:

$$J = -\int_{-\pi a/4}^{\pi a/4} \rho g y \, ds + \lambda \int_{-\pi a/4}^{\pi a/4} (1 - y'^2)^{\frac{1}{2}} \, ds$$

In term of Euler Lagrange eqns.

$$F = -\rho g y + \lambda (1 - y'^2)^{\frac{1}{2}}$$

$F(y, y', s)$: s does not appear, therefore use second special case

$$F - y' \frac{\partial F}{\partial y'} = k \quad (k: \text{constant})$$

$$-\rho g y + \lambda (1 - y'^2)^{\frac{1}{2}} - y' \lambda \left(\tfrac{1}{2}\right)(1 - y'^2)^{-\frac{1}{2}}(-2y')$$

$y' = $

$$\rightarrow -\rho g y + \lambda (1 - y'^2)^{\frac{1}{2}} + y'^2 \lambda (1 - y'^2) = k$$

$$-\rho g y (1 - y'^2)^{\frac{1}{2}} + \lambda (1 - y'^2) + \lambda y'^2 = k$$

$$\rightarrow -\rho g y (1 - y'^2)^{\frac{1}{2}} + \lambda = k$$

$$\rightarrow \rho = \frac{\cancel{k} \; \lambda - k}{g y (1 - y'^2)^{\frac{1}{2}}} \qquad (1)$$

Since string is on arc,

$$y = y_0 + a \cos(s/a)$$

Verify that for this y that $\rho \propto \sec^2(s/a)$

2c) cont.

$$y' = -\sin(s/a)$$

$$\rho = \frac{\lambda - k}{g(y_0 + a\cos(s/a))(1-\sin^2(s/a))^{\frac{1}{2}}}$$

$$= \frac{\lambda - k}{g(y_0 + a\cos(s/a))\cos(s/a)}$$

Set $y_0 = 0$

$$\rho = \frac{\lambda - k}{ga\cos^2(s/a)}$$

$$\rho \propto 1/\cos^2(s/a) = \sec^2(s/a)$$

$$\phi = R(r)\,H(\theta)$$

(a) $$\left[\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2}\right] = 0$$

$$\Rightarrow R''(r)\,H(\theta) + r^{-1}R'(r)\,H(\theta) + r^{-2}R(r)\,H''(\theta) = 0$$

$$\Rightarrow \frac{r^2 R''(r) + rR'(r)}{R(r)} = -\frac{H''(\theta)}{H(\theta)} = k$$

In this equation LHS is independent of θ, RHS is independent of r. Hence both sides must be a constant k.

(b) The boundary conditions are

on $r = a$ for $0 < \theta < \pi$, $R'(a)\,H(\theta) = 0 \Rightarrow R'(a) = 0$

on $\theta = 0$ for $r > a$, $R(r)\,H'(0) = 0 \Rightarrow H'(0) = 0$

on $\theta = \pi$ for $r > a$, $R(r)\,H'(\pi) = 0 \Rightarrow H'(\pi) = 0$

as $r \to \infty$, $R(r)\,H(\theta) = (x^2 - y^2) = R(r)\,H(\theta) - r^2\cos 2\theta = 0$

(c) as $k > 0$ (* proved at the end), we take the constant as n^2. the equation for H is

$$H''(\theta) + n^2 H(\theta) = 0$$

$$\Rightarrow H(\theta) = A\cos(n\theta) + B\sin(n\theta)$$

$$\Rightarrow H'(\theta) = -An\sin(n\theta) + Bn\cos(n\theta)$$

B.C $H'(0) = 0 \Rightarrow B = 0$, $\quad H'(\pi) = 0 \Rightarrow \sin(n\theta) = 0$

$$\Rightarrow \boxed{n = \text{non-zero integer}}$$

Hence the solution for H are $\boxed{\text{Costant} \times \cos(n\theta)}$

(d) Return to the equation for $R(r)$:

$$\begin{cases} r^2 R''(r) + r R'(r) - n^2 R(r) = 0 \\ R'(a) = 0 \end{cases}$$

The equation is homogeneous type, has solution like r^α,

where $r^2 \alpha(\alpha-1) r^{\alpha-2} + r\alpha r^{\alpha-1} - n^2 r^\alpha = 0$

$$\Rightarrow \quad \alpha = \pm n$$

$$\Rightarrow \quad R(r) = C r^n + D r^{-n}.$$

To fit the B. C. $R'(a) = 0$,

$$R(r) = \text{constant} \times \left[\left(\frac{r}{a}\right)^n + \left(\frac{a}{r}\right)^n \right].$$

(e). We have reached the stage that

$$\phi(r,\theta) = \sum_{n=1}^{\infty} A_n \left[\left(\frac{r}{a}\right)^n + \left(\frac{a}{r}\right)^n \right] \cos(n\theta)$$

where A_n is constant.

To fit the boundary condition $\phi - r^2\cos 2\theta = O(1)$, as $r \to \infty$

$$\boxed{\begin{array}{ll} A_n = 0 & \text{for } n \neq 2 \\ A_2 = a^2 & \end{array}}$$

$$\boxed{\phi(r,\theta) = a^2 \left[\left(\frac{r}{a}\right)^2 + \left(\frac{a}{r}\right)^2 \right] \cos(2\theta)}$$

(c) ① If $k=0$, $H''(\theta)=0$

$$\Rightarrow \quad H = C_0 + C_1\theta$$

$$\left.\begin{aligned} H'(\theta) &= C_1 \\ \text{B.C. } H'(0) &= 0 \end{aligned}\right\} \Rightarrow C_1 \Rightarrow H(\theta) = C_0 = \text{constant}$$

not good.

② If $k<0$, take $k=-n^2$.

$$H'' = n^2 H \Rightarrow \quad H(\theta) = Ae^{n\theta} + Be^{-n\theta}$$

$$\left.\begin{aligned} H'(\theta) &= n(Ae^{n\theta} - Be^{-n\theta}) \\ H'(0) &= 0 \end{aligned}\right\} \Rightarrow A = B$$

So $\quad H(\theta) = A\,(e^{n\theta} + e^{-n\theta}) = 2A\sinh(n\theta)$

$$\left.\begin{aligned} H'(\theta) &= 2An\cosh(n\theta) \\ H'(\pi) &= 0 \end{aligned}\right\} \Rightarrow \text{impossible.}$$

Therefore, we are left to the case $k>0$, take $k=n^2$.

(a) Define $\underline{\underline{A}} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$

The system is hyperbolic if the two eigenvalues λ_1, λ_2 of the characteristic determinant

$$|\underline{\underline{A}} - \lambda \underline{\underline{I}}| = 0$$

are real and distinct. Two corresponding left eigenvectors $\underline{l}^{(i)} = 1, 2$:

$$\underline{l}^{(i)} \underline{\underline{A}} = \lambda^{(i)} \underline{l}^{(i)} .$$

The two families of characteristic curves $C^{(i)}$ are defined by $C^{(i)}: \dfrac{dx}{dt} = \lambda^{(i)}$ for $i = 1, 2$.

The left eigenvectors are two row vectors $\underline{l}^{(i)} = (l_1^{(i)}, l_2^{(i)})$.

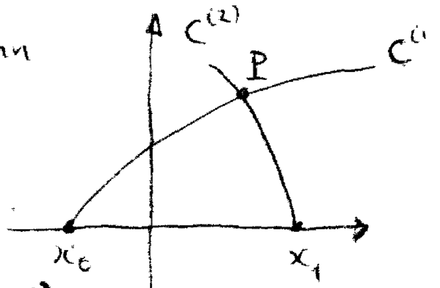
$$l_1^{(i)} \frac{du}{dt} + l_2^{(i)} \frac{dv}{dt} = 0 \quad \text{along } C^{(i)} \text{ characteristics}$$

There exist integrating factors $\mu^{(i)}(u,v)$ with the property that the above equations can be integrated:

$$\int \mu^{(i)} l_1^{(i)} \, du + \int \mu^{(i)} l_2^{(i)} \, du_2 = \underline{\text{constant} = R^{(i)}}$$

The constant at the right hand side is called Riemann invariant.

Consider the figure at left. The Riemann invariant $R^{(i)}$ is determined by the I.C. $R^{(1)} = R^{(1)}(x_0, 0)$, while $R^{(2)} = R^{(2)}(x_1, 0)$. The (u,v) at point P is then determined by the known values $R^{(1)}(u,v)$ and $R^{(2)}(u,v)$



(b) The matrix $\underline{\underline{A}} = \begin{pmatrix} 0 & 1 \\ a^2 - \frac{m^2}{\rho^2} & \frac{2m}{\rho} \end{pmatrix}$

The eigenvalues are

$$\boxed{\lambda_1 = \frac{m}{\rho} - a \quad , \quad \lambda_2 = \frac{m}{\rho} + a}$$

left eigenvector for λ_1

$$(l_1, l_2) \begin{pmatrix} 0 & 1 \\ a^2 - \frac{m^2}{\rho^2} & \frac{2m}{\rho} \end{pmatrix} = \left(\frac{m}{\rho} - a\right)(l_1, l_2)$$

$$l_2 \left(a^2 - \frac{m^2}{\rho^2}\right) = \left(\frac{m}{\rho} - a\right) l_1$$

$$l_1 = -\left(a + \frac{m}{\rho}\right) l_2$$

$\Rightarrow$ left eigenvector $\boxed{\left(-a - \frac{m}{\rho} , 1\right)}$

similarly, left eigenvector for λ_2, $\boxed{\left(a - \frac{m}{\rho} , 1\right)}$

(c) The system is

$$\begin{pmatrix} \frac{\partial \rho}{\partial t} \\ \frac{\partial m}{\partial t} \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ a^2 - \frac{m^2}{\rho^2} & \frac{2m}{\rho} \end{pmatrix} \begin{pmatrix} \frac{\partial \rho}{\partial x} \\ \frac{\partial m}{\partial x} \end{pmatrix} = 0$$

Multiply ~~$\lambda_1 = \frac{m}{\rho} - a$~~ the left eigenvector of λ_1

$$\left(-a - \frac{m}{\rho}\right) \frac{\partial \rho}{\partial t} + \frac{\partial m}{\partial t} + \left(a^2 - \frac{m^2}{\rho^2}\right) \frac{\partial \rho}{\partial x} + \left(-a + \frac{m}{\rho}\right) \frac{\partial m}{\partial x} = 0$$

$$\Rightarrow \quad -\left(a + \frac{m}{\rho}\right) \frac{d\rho}{dt} + \frac{dm}{dt} = 0 \quad \text{on } C^{(1)} : \frac{dx}{dt} = \frac{m}{\rho} - a$$

$$-\frac{a}{\rho} \frac{d\rho}{dt} + \underbrace{\left(\frac{1}{\rho} \frac{dm}{dt} - \frac{m}{\rho^2} \frac{d\rho}{dt}\right)}_{\frac{d}{dt}\left(\frac{m}{\rho}\right)} = 0$$

$$\Rightarrow +\frac{d}{dt}\left(a\log\rho\right) + \frac{m}{\rho}\Big) = 0$$

$$\Rightarrow \frac{m}{\rho} - a\log\rho = \cancel{\text{Constaion}}$$

$$= \text{constant} = r, \text{ on } C^{(1)}.$$

$$\boxed{\frac{m}{\rho} - a\log\rho = r}$$

Similarly, multiply the left eigenvector of λ_2,

$$\left(a - \frac{m}{\rho}\right)\frac{d\rho}{dt} + \frac{dm}{dt} = 0 \quad \text{on } C^{(2)}: \frac{dx}{dt} = \frac{m}{\rho} +$$

$$\frac{a}{\rho}\frac{d\rho}{dt} + \left(\frac{1}{\rho}\frac{dm}{dt} - \frac{1}{\rho^2}\frac{d\rho}{dt}\right) = 0$$

$$\Rightarrow \frac{d}{dt}\left(a\log\rho + \frac{m}{\rho}\right) = 0$$

$$\Rightarrow \boxed{\frac{m}{\rho} + a\log\rho = s} = \text{constant on } C^{(2)}.$$

Solving ρ and m in terms of the Riemann invariants r and s, we find that

$$\boxed{\rho = \exp\left(\frac{s-r}{2a}\right)},$$

$$\frac{m}{\rho} = \frac{r+s}{2} \Rightarrow \boxed{m = \left(\frac{r+s}{2}\right)e^{\left(\frac{s-r}{2a}\right)}}$$

Examiner's comments:

Q1. Index notation, integration by parts and Lagrange multipliers

Most candidates managed most of parts (a) and (b). Some attempted to incorrectly use Stokes Theorem (confusing this with Stokes equations). Few candidates completed part (c) successfully, with many failing to use a Lagrange multiplier to impose the constraint and/or overcomplicating the problem by taking variations with respect to f, which is fixed. Few identified the pressure as playing the role of a Lagrange multiplier.

Q2. Constrained minimisation

Almost all candidates answered part (a) correctly and formulated the potential energy expression in part (b). Numerous candidates formulated a trivial constraint for part (b), but failed to express it in terms of y. A number of candidates incorrectly took variations with respect to ρ in part (c).

Q3. Separation of variables

Almost all candidates could apply separation of variables, but many did so to the wrong system, failing to convert the Laplace operator into polar coordinates. It was often treated in Cartesian coordinates, and in some cases spherical coordinates.

Q4. Characteristics for systems of first-order PDEs

Most candidates successfully formulated the hyperbolic condition. However, a number introduced the determinant of the matrix A, rather than considering eigenvalues of A. Most could find the eigenvalues and eigenvectors for part (b). Most candidates struggled with part (c).

G.N. Wells (Principal Assessor)