ENGINEERING TRIPOS PART IIB 2012
4M13 COMPLEX ANALYSIS AND OPTIMIZATION Prof N A Fleck

Module 4M13 : Complex Analysis & Optimization.

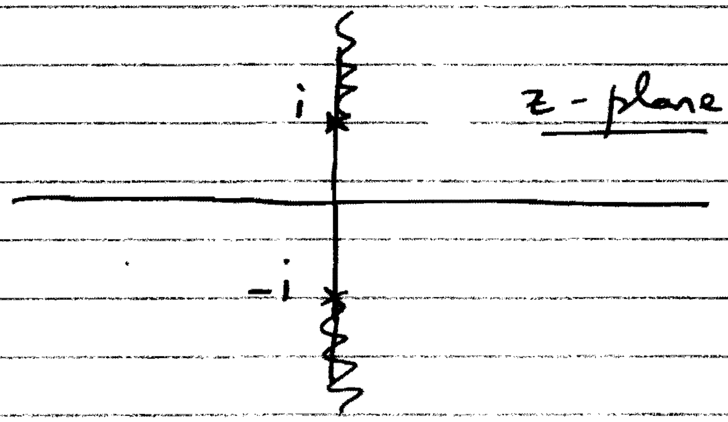
1 (a) (i) $f(z) = z^2 \ln(z^2+1)^{1/3}$

$\Rightarrow f(z) = \frac{z^2}{3} \ln[(z+i)(z-i)]$

$\Rightarrow$ a zero at $z = 0$, branch points at $z = \pm i$

As $z \to \infty$, $f(z) \to \frac{2}{3} z^2 \ln z$

So the branch cut extends out to ∞



(ii) $f(z) = \frac{z-i}{z^2+iz+2}$

Now, $z^2 + iz + 2 = (z-i)(z+2i)$

$\Rightarrow f(z) = \frac{1}{z+2i}$

So, $f(z)$ has a simple pole at $z = -2i$

Residue is $\lim_{z \to -2i} [(z+2i) f(z)] = 1$.

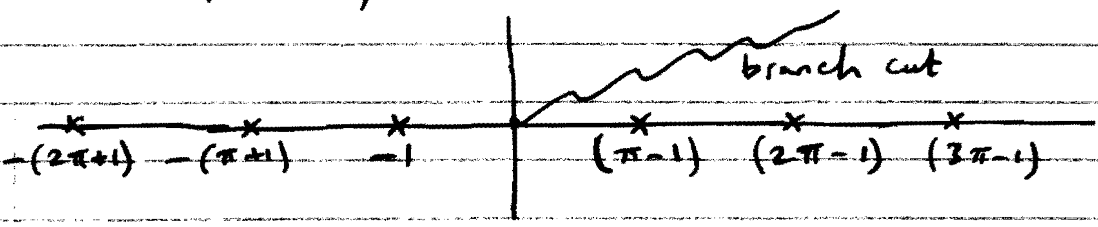
Crib for paper 4 M13, 2012

1 (a) (iii) $f(z) = \dfrac{z^{1/3}}{\sin\left(\dfrac{z^2-1}{z-1}\right)}$

$$f(z) = \frac{z^{1/3}}{\sin(z+1)}$$

Branch points at $z = 0, \infty$

simple poles of $f(z)$ exist



Note that $\sin(z+1) = 0$ at $z = n\pi - 1$, where n is an integer.

$\Rightarrow$ $f(z)$ has simple poles at $z = n\pi - 1$.

Residue at the pole $z = n\pi - 1$ is

$$(n\pi - 1)^{1/3} \; (-1)^n$$

1. (b) Calculate

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} \frac{\omega+1}{(\omega-2i)^2}\, d\omega = ?$$

Consider the integrand $I(\omega) = \dfrac{\omega+1}{(\omega-2i)^2} e^{i\omega t}$

This has a pole of order 2 at $\omega = 2i$.

Crib for paper 4M13

1. (b) cont'd.

What is the residue of $I(\omega)$ at $\omega = 2i$? To determine this, expand the top line of $I(\omega)$ by a Taylor Series Expansion, to give

$$(\omega+1)\, e^{i\omega t} \approx \; ?$$

Write $g(\omega) = (\omega+1)\, e^{i\omega t}$. Then near $\omega = 2i$ we have

$$g(\omega) \approx g(2i) + (\omega - 2i)\, g'(2i) + \cdots$$

Now, $g(\omega = 2i) = (1+2i)\, e^{-2t}$

$$g'(\omega) = e^{i\omega t} + (\omega+1)\, it\, e^{i\omega t}$$

and

$$g'(\omega = 2i) = e^{-2t} + (2i+1)\, it\, e^{-2t}$$

So, $g(\omega) \approx (1+2i)\, e^{-2t} + (\omega - 2i)\left[e^{-2t} + (2i+1)\, it\, e^{-2t}\right]$

and the residue of $I(\omega)$ is

$$\text{Res} = e^{-2t} + (2i+1)\, it\, e^{-2t}$$

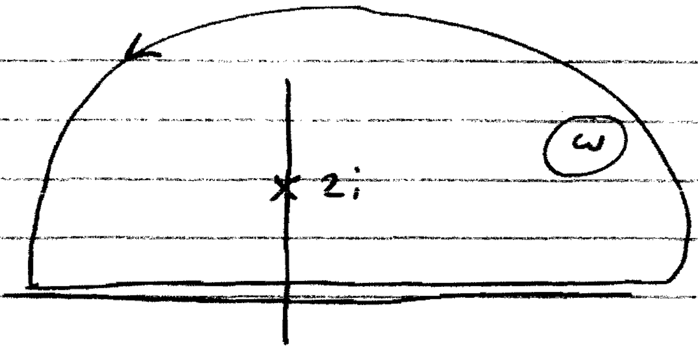
$$= e^{-2t}\left[1 + it(2i+1)\right]$$

Back to the integral $\frac{1}{2\pi}\int_{-\infty}^{\infty} I(\omega)\, d\omega$

$t < 0$: $f(t) = 0$ – close the integral in the L.H.P. by Jordan's Lemma with no poles in the L.H.P.

1. (b) contd.

$t > 0$



Now close the contour in the UHP. and invoke Jordans Lemma : no contribution to the line integral on the semi-circular contour at ∞.

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} I(\omega)\, d\omega = \frac{1}{2\pi}\oint I(\omega)\, d\omega$$

$$= \frac{1}{2\pi} \times 2\pi i \times \text{Sum of Residues of } I(\omega)$$

$$= i \,.\, e^{-2t}\left[1 + it(2i+1)\right]$$

Examiner's comments:

A popular and straightforward question, well-answered by most candidates. Several candidates struggled with part 1(a)(iii) which had both poles and branch points. Most made appropriate use of Jordan's Lemma and shoed a good overall understanding of how to classify singularities and perform contour integration.

Crib for 4M13

Q 2 (a)

$$f(z) = \frac{\ln z}{(z-1)^2}$$

Write $g(z) = \ln z$ and expand about the point z_0 by a Taylor expansion :

$$g(z) \approx g(z_0) + (z-z_0)\, g'(z_0) + \frac{(z-z_0)^2}{2!} g''(z_0) \cdot$$

Here, $z_0 = 1 \Rightarrow g(z_0) = g(1) = \ln(1) = 0$

$$g'(z) = \frac{1}{z} \Rightarrow g'(1) = 1$$

$$g''(z) = \frac{-1}{z^2} \Rightarrow g''(1) = -1$$

$$\Rightarrow g(z) \approx (z-1) + \frac{(z-1)^2}{2!}(-1) + \ldots$$

$$\Rightarrow f(z) \approx \frac{1}{z-1} - \frac{1}{2} + \ldots$$

$\Rightarrow$ Residue of $f(z)$ is 1 at $z=1$.

Note $f(z)$ has a simple pole at $z=1$, not a pole of order 2.

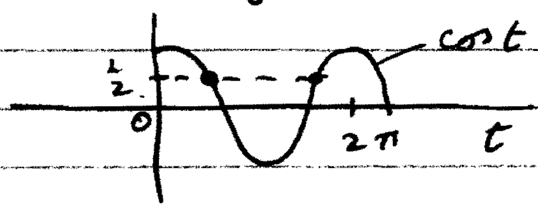
Crib for 4M13

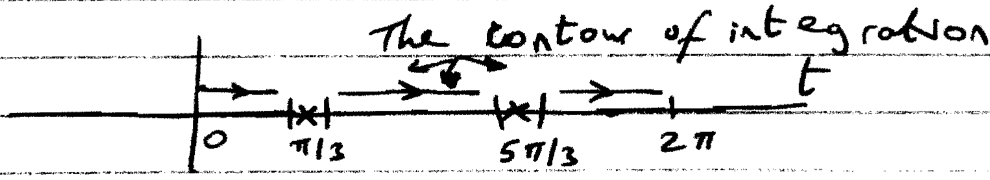
2. (b) $I = \text{P.V.} \int_0^{2\pi} \frac{2}{(2\cos t - 1)} \, dt = ?$

Principal value ↗ (P.V.)

Note: integrand is singular at $\cos t = 1/2$

i.e. at $t = \frac{\pi}{3}, \frac{5\pi}{3}$





So, the integrand has simple poles at $t = \pi/3, \frac{5\pi}{3}$ on the integration path. Hence, the need for defining a principal value.

Now change variables: $z = e^{it}$

$\Rightarrow \quad dz = iz \, dt$

As t increases from 0 to 2π, z moves around the unit circle in the z-plane.

$$\cos t = \tfrac{1}{2}(e^{it} + e^{-it}) = \tfrac{1}{2}\left(z + \tfrac{1}{z}\right)$$

$$\Rightarrow \quad I = \text{P.V.} \oint \frac{2}{iz} \frac{1}{(z + \frac{1}{z} - 1)} \, dz$$

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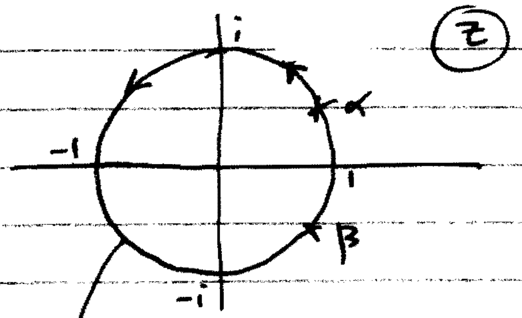
2. (b) contd.

$$\Rightarrow \quad I = \text{P.V.} \ \frac{2}{i} \oint_C \frac{1}{z^2 - z + 1} \, dz$$

Now, write $(z^2 - z + 1)$ as $(z-\alpha)(z-\beta)$

where $\alpha = \dfrac{1 + i\sqrt{3}}{2}$, $\beta = \dfrac{1 - i\sqrt{3}}{2}$

$$\Rightarrow \quad I = \text{P.V.} \oint_C f(z)\, dz \quad \text{where} \quad f(z) = \frac{2}{i} \frac{1}{(z-\alpha)(z-\beta)}$$



Contour C is unit circle

$I = \pi i \times$ sum of residues of $f(z)$ **on** path of integration.
↑
not $2\pi i$

Residue of $f(z)$ at $z = \alpha$ is $\dfrac{2}{i} \dfrac{1}{\alpha - \beta} = \dfrac{-2}{\sqrt{3}}$

Residue of $f(z)$ at $z = \beta$ is $\dfrac{2}{i} \dfrac{1}{\beta - \alpha} = \dfrac{2}{\sqrt{3}}$

$\Rightarrow$ Sum of residues $= 0$

$\Rightarrow$ $I = 0$.

Examiner's comment

A popular question. Almost all know how to determine a Laurent Series and residues. Most understood the main ideas of Principal Values but many did not realize that the singularities were not on the contour of integration. Most struggled to evaluate the residues in part (b) correctly.

Question 3

(a) From Fig. 1:
$$L^2 = h^2 + \left(\frac{s}{2}\right)^2$$

$$\therefore \quad L^2 = 1^2 + \left(\frac{1.5}{2}\right)^2 = 1.5625 \quad \Rightarrow \quad L = 1.25 \text{ m}$$

Also from Fig. 1:
$$\cos\beta = \frac{s}{2L} = \frac{1.5}{2\times 1.25} = 0.6$$

$$\sin\beta = \frac{h}{L} = \frac{1}{1.25} = 0.8$$

$$P(x,y) = \frac{EA}{2L}\left[(x\cos\beta + y\sin\beta)^2 + (-x\cos\beta + y\sin\beta)^2\right] - W[x\cos\theta + y\sin\theta]$$

$$\therefore \quad P(x,y) = \frac{EA}{2L}\left[2x^2\cos^2\beta + 2y^2\sin^2\beta\right] - W[x\cos\theta + y\sin\theta]$$

$$\therefore \quad P(x,y) = \frac{200\times 10^9 \times 10^{-5}}{2\times 1.25}\left[2x^2 \times 0.6^2 + 2y^2 \times 0.8^2\right] - 10^4[x\cos 30 + y\sin 30]$$

$$\therefore \quad P(x,y) = 5.76\times 10^5 x^2 + 1.024\times 10^6 y^2 - 8.66\times 10^3 x - 5\times 10^3 y$$ [15%]

(b) The common scaling factor of 10^3 can be eliminated without changing the nature of the problem:

$$\therefore \quad P(x,y) = 576x^2 + 1024y^2 - 8.66x - 5y$$

$$\therefore \quad \frac{\partial P}{\partial x} = 1152x - 8.66 \qquad (1)$$

$$\therefore \quad \frac{\partial^2 P}{\partial x^2} = 1152 \qquad (2)$$

$$\therefore \quad \frac{\partial P}{\partial y} = 2048y - 5 \qquad (3)$$

$$\therefore \quad \frac{\partial^2 P}{\partial y^2} = 2048 \qquad (4)$$

$$\therefore \quad \frac{\partial^2 P}{\partial x \partial y} = 0 \qquad (5)$$

From (1): $\frac{\partial P}{\partial x} = 1152x - 8.66 = 0 \quad \Rightarrow \quad x = \frac{8.66}{1152} = 7.52\times 10^{-3}$ m

From (3): $\frac{\partial P}{\partial y} = 2048y - 5 = 0 \quad \Rightarrow \quad y = \frac{5}{2048} = 2.44\times 10^{-3}$ m

From (2), (4) and (5) the Hessian: $\boldsymbol{H} = \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix}$

By inspection $\boldsymbol{H}$ is positive definite, so this solution is a minimum. [20%]

(c) Let $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$

$$f(\mathbf{x}_{k+1}) = f(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = f(\mathbf{x}_k) + \alpha_k \nabla f(\mathbf{x}_k)^T \mathbf{d}_k + \frac{\alpha_k^2}{2} \mathbf{d}_k^T \boldsymbol{H}(\mathbf{x}_k)\mathbf{d}_k + R$$

Neglecting R and differentiating with respect to α_k:

$$\frac{\partial f}{\partial \alpha_k}(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = \nabla f(\mathbf{x}_k)^T \mathbf{d}_k + \alpha_k \mathbf{d}_k^T \boldsymbol{H}(\mathbf{x}_k)\mathbf{d}_k$$

For a minimum:

$$\frac{\partial f}{\partial \alpha_k}(\mathbf{x}_k + \alpha_k \mathbf{d}_k) = \nabla f(\mathbf{x}_k)^T \mathbf{d}_k + \alpha_k \mathbf{d}_k^T \boldsymbol{H}(\mathbf{x}_k)\mathbf{d}_k = 0$$

$$\therefore \quad \alpha_k = -\frac{\nabla f(\mathbf{x}_k)^T \mathbf{d}_k}{\mathbf{d}_k^T \boldsymbol{H}(\mathbf{x}_k)\mathbf{d}_k}$$ [15%]

(d) From (b): $\nabla f(x,y) = \begin{bmatrix} 1152x - 8.66 \\ 2048y - 5 \end{bmatrix}$

For the SDM: $\mathbf{d} = -\nabla f = \begin{bmatrix} 8.66 - 1152x \\ 5 - 2048y \end{bmatrix}$

$$\therefore \quad \alpha_k = \frac{\mathbf{d}_k^T \mathbf{d}_k}{\mathbf{d}_k^T \boldsymbol{H}(\mathbf{x}_k)\mathbf{d}_k}$$

From (b): $\boldsymbol{H} = \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix}$

If $\mathbf{x}_1 = (0,0)$: $\mathbf{d}_1 = \begin{bmatrix} 8.66 \\ 5 \end{bmatrix}$

$$\therefore \quad \alpha_1 = \frac{8.66^2 + 5^2}{\begin{bmatrix} 8.66 & 5 \end{bmatrix} \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix} \begin{bmatrix} 8.66 \\ 5 \end{bmatrix}}$$

$$\therefore \quad \alpha_1 = \frac{8.66^2 + 5^2}{1152 \times 8.66^2 + 2048 \times 5^2} = 7.267 \times 10^{-4}$$

$$\therefore \quad \mathbf{x}_2 = \mathbf{x}_1 + \alpha_1 \mathbf{d}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 7.267 \times 10^{-4} \begin{bmatrix} 8.66 \\ 5 \end{bmatrix} = \begin{bmatrix} 6.294 \times 10^{-3} \\ 3.634 \times 10^{-3} \end{bmatrix}$$

$$\therefore \quad \mathbf{d}_2 = \begin{bmatrix} 8.66 - 1152 \times 6.294 \times 10^{-3} \\ 5 - 2048 \times 3.634 \times 10^{-3} \end{bmatrix} = \begin{bmatrix} 1.409 \\ -2.442 \end{bmatrix}$$

$$\therefore \quad \alpha_2 = \frac{1.409^2 + (-2.442)^2}{\begin{bmatrix} 1.409 & -2.442 \end{bmatrix} \begin{bmatrix} 1152 & 0 \\ 0 & 2048 \end{bmatrix} \begin{bmatrix} 1.409 \\ -2.442 \end{bmatrix}}$$

$$\therefore \quad \alpha_2 = \frac{1.409^2 + 2.442^2}{1152 \times 1.409^2 + 2048 \times 2.442^2} = 5.482 \times 10^{-4}$$

$$\therefore \quad \mathbf{x}_3 = \mathbf{x}_2 + \alpha_2 \mathbf{d}_2 = \begin{bmatrix} 6.294 \times 10^{-3} \\ 3.634 \times 10^{-3} \end{bmatrix} + 5.482 \times 10^{-4} \begin{bmatrix} 1.409 \\ -2.442 \end{bmatrix} = \begin{bmatrix} 7.066 \times 10^{-3} \\ 2.295 \times 10^{-3} \end{bmatrix}$$ [35%]

(e) The Steepest Descent Method is making good progress towards the minimum. By inspection the eigenvalues of the Hessian are 1152 and 2048. Therefore the convergence ratio β is bounded by:

$$\beta \le \left[\frac{A - a}{A + a}\right]^2 = \left[\frac{2048 - 1152}{2048 + 1152}\right]^2 = 0.0784$$

This comparatively small value of β explains the good convergence.

The problem is quadratic so Newton's Method will converge in one iteration, and the Conjugate Gradient Method will converge in a number of iterations equal to the number of control variables, i.e. two iterations in this case. [15%]

Examiner's comment:

A very popular question that was well done by many candidates. The most common sources of error were: a failure to check the second-order optimality conditions in part (b); and numerical slips in executing the Steepest Descent Method (SDM). Part (c), which asked for a standard derivation, was done surprisingly badly. Not many candidates recognised that they could straightforwardly calculate the upper bound on the SDM convergence ratio in part (e) to explain its performance.

Question 4

(a) The total cost of installation to be minimized is:

$$f(D,L)=150D^2L+25D^{2.5}L+20DL$$

subject to $$T=15D^2L\geq 100$$

$$\therefore\quad g_1=100-15D^2L\leq 0$$

In principle, there are non-negativity bounds on D and L, but it is obvious that they cannot be active if g_1 is to be satisfied, and can therefore be omitted. [10%]

(b) Assuming the constraint on T is active, the problem is:

Minimize $$f(D,L)=150D^2L+25D^{2.5}L+20DL$$

subject to $$h_1=100-15D^2L=0$$

The Lagrangian is $l=150D^2L+25D^{2.5}L+20DL+\lambda\left(100-15D^2L\right)$

$$\therefore\quad \frac{\partial l}{\partial D}=300DL+62.5D^{1.5}L+20L-30\lambda DL=0 \qquad (1)$$

$$\frac{\partial l}{\partial L}=150D^2+25D^{2.5}+20D-15D^2\lambda=0 \qquad (2)$$

$$100-15D^2L=0 \qquad (3)$$

From (1) $$300D+62.5D^{1.5}+20-30\lambda D=0 \qquad (4)$$

From (2) $$15D^2\lambda=150D^2+25D^{2.5}+20D$$

$$\therefore\quad 30\lambda D=300D+50D^{1.5}+40$$

Substituting in (4) $$300D+62.5D^{1.5}+20-300D-50D^{1.5}-40=0$$

$$\therefore\quad 12.5D^{1.5}=20 \quad\Rightarrow\quad D=1.368\text{ m}$$

From (3) $$L=\frac{100}{15D^2}$$

$$\therefore\quad L=\frac{100}{15D^2}=\frac{100}{15(1.368)^2}=3.562\text{ m}$$ [40%]

(c) With the introduction of the new constraint we can no longer assume that the constraint on T is active, therefore we have now have a problem with two inequality constraints:

Minimize $$f(D,L)=150D^2L+25D^{2.5}L+20DL$$

Subject to $$g_1=100-15D^2L\leq 0$$

and $$g_2=DL-4\leq 0$$

The Lagrangian is now:

$$l=150D^2L+25D^{2.5}L+20DL+\mu_1\left(100-15D^2L\right)+\mu_2\left(DL-4\right)$$

$$\therefore \quad \frac{\partial \mathcal{L}}{\partial D} = 300DL + 62.5D^{1.5}L + 20L - 30\mu_1 DL + \mu_2 L = 0 \qquad (1)$$

$$\frac{\partial \mathcal{L}}{\partial L} = 150D^2 + 25D^{2.5} + 20D - 15D^2\mu_1 + \mu_2 D = 0 \qquad (2)$$

$$\mu_1\left(100 - 15D^2L\right) = 0 \qquad (3)$$

$$\mu_2\left(DL - 4\right) = 0 \qquad (4)$$

Case (i) $\mu_1 = 0$ and $\mu_2 = 0$

$(2) \Rightarrow \quad 150D^2 + 25D^{2.5} + 20D = 0$

$\therefore \quad D = 0$ (impossible)

or $\quad 150D + 25D^{1.5} + 20 = 0 \quad \Rightarrow \quad$ no real non-negative solution for D

$\therefore$ **impossible**

Case (ii) $\mu_1 = 0$ and $\mu_2 > 0$

$(2) \Rightarrow \quad 150D^2 + 25D^{2.5} + 20D + \mu_2 D = 0$

$\therefore \quad D = 0$ (impossible)

or $\quad \mu_2 = -150D - 25D^{1.5} - 20 \quad \Rightarrow \quad \mu_2 < 0$

$\therefore$ **not a minimum**

Case (iii) $\mu_1 > 0$ and $\mu_2 = 0$

This is equivalent to the case solved in part (b) where the constraint on T is active and the constraint on A is inactive (in (b) it did not apply).

For this case we know that $D = 1.368$ m and $L = 3.562$ m.

We need to check that $g_2 = DL - 4 \le 0$ is not violated:

$g_2 = 1.368 \times 3.562 - 4 = 0.873 \quad \therefore\ g_2$ is violated

$\therefore$ **impossible**

Case (iv) $\mu_1 > 0$ and $\mu_2 > 0$

$(3) \Rightarrow \quad 100 - 15D^2L = 0$

$\therefore \quad 15D^2L = 100$

$(4) \Rightarrow \quad DL - 4 = 0 \quad \Rightarrow \quad L = \frac{4}{D}$

$\therefore \quad 15D^2\frac{4}{D} = 100 \quad \Rightarrow \quad D = \frac{5}{3}\text{ m} \quad \Rightarrow \quad L = 4 \times \frac{3}{5} = 2.4\text{ m}$

$(1) \Rightarrow \quad 300DL + 62.5D^{1.5}L + 20L - 30\mu_1 DL + \mu_2 L = 0$

$\therefore \quad 300D + 62.5D^{1.5} + 20 - 30\mu_1 D + \mu_2 = 0$

$$\therefore \quad 300\left(\frac{5}{3}\right)+62.5\left(\frac{5}{3}\right)^{1.5}+20-30\mu_1\left(\frac{5}{3}\right)+\mu_2=0$$

$$\therefore \quad 654.48-50\mu_1+\mu_2=0 \qquad (5)$$

$$(2)\Rightarrow \quad 150D^2+25D^{2.5}+20D-15D^2\mu_1+\mu_2 D=0$$

$$\therefore \quad 150D+25D^{1.5}+20-15D\mu_1+\mu_2=0$$

$$\therefore \quad 150\left(\frac{5}{3}\right)+25\left(\frac{5}{3}\right)^{1.5}+20-15\left(\frac{5}{3}\right)\mu_1+\mu_2=0$$

$$\therefore \quad 323.79-25\mu_1+\mu_2=0 \qquad (6)$$

$$(5)-(6)\Rightarrow \quad 330.69-25\mu_1=0 \quad \Rightarrow \quad \mu_1=13.228$$

$$\therefore \quad \mu_2=6.9$$

As $\mu_1>0$ and $\mu_2>0$ $\therefore$ **a minimum**

Thus, the new optimal design is $D=1.667$ m and $L=2.4$ m. [50%]

Examiner's comment:

Another very popular question, but not done as well as Q3. The average mark was also dragged down by the fact that the question that attracted a few very scratchy partial attempts. Most candidates showed that they had a good idea of how the Lagrange and Kuhn-Tucker Multiplier methods work. Algebraic mistakes were the most common cause of failure to answer part (b) (about Lagrange Multipliers). A number of candidates claimed dishonestly that the quoted solution solved their incorrect equations. This was penalised more harshly than answers where the candidate admitted that something must have gone wrong. Common problems in answering part (c) (about Kuhn-Tucker multipliers) were: not testing that the multipliers were positive at potential optima; not recognising that the solution to part (b) corresponded to one of the cases that needed testing; and failing to spot when cases could not correspond to a minimum because a multiplier would inevitably be negative for any physically plausible (i.e. non-negative) values of the control variables.