ENGINEERING TRIPOS PART IIB

Thursday 3 May 2012 2.30 to 4

Module 4A10

FLOW INSTABILITY

Answer not more than ***three*** *questions.*

All questions carry the same number of marks.

The ***approximate*** *percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments: 4A10 data sheet (two pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

mpj05

1 Figure 1 shows an inviscid, incompressible plane liquid jet of density ρ flowing between free surfaces at $z = \pm a$ at a uniform speed U in the positive x-direction. The surface tension between the air and the liquid is σ. Assume that the jet is unbounded in the x and y directions. Choose a reference frame in which the liquid is initially at rest and assume that the air has negligible density in comparison with the liquid. The stability of the liquid is governed by the equation

$$\nabla^2 p' = 0 \text{ for } -\infty < x, y < \infty, -a \le z \le a$$

where p' denotes the pressure perturbation.

(a) Show that the boundary conditions are given by

$$\rho \frac{\partial^2 p'}{\partial t^2} = \pm\sigma \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \frac{\partial p'}{\partial z} \text{ at } z = \pm a$$

[30%]

[*Hints:* $\rho \partial \boldsymbol{u}'/\partial t = -\nabla p'$, $\nabla \cdot \boldsymbol{u}' = 0$, and on $z = \pm a$, $u_z' = \partial \zeta'/\partial t$, $p' = \sigma \nabla \cdot \boldsymbol{n}$, and $\boldsymbol{n} = \pm(-\partial \zeta'/\partial x, -\partial \zeta'/\partial y, 1)$, where $\boldsymbol{u}'$ denotes the velocity perturbation and ζ' denotes the surface displacement in the z direction. Note that the z derivatives at the boundary are evaluated on the liquid side of the interface.]

(b) Taking normal modes with $p' = f(z)\exp\{st + \mathrm{i}(kx + ly)\}$, show that:

$$f(z) = A\cosh\left(\sqrt{k^2 + l^2}\, z\right) + B\sinh\left(\sqrt{k^2 + l^2}\, z\right)$$

where A and B are constants. [20%]

(c) Hence deduce that for the even ($B = 0$) mode,

$$a^3 \rho s^2/\sigma = -\tilde{\alpha}^3 \tanh \tilde{\alpha}$$

and for the odd ($A = 0$) mode,

$$a^3 \rho s^2/\sigma = -\tilde{\alpha}^3 \coth \tilde{\alpha}$$

where $\tilde{\alpha} = a\sqrt{k^2 + l^2}$. [30%]

(d) Deduce the stability of the jet. [20%]

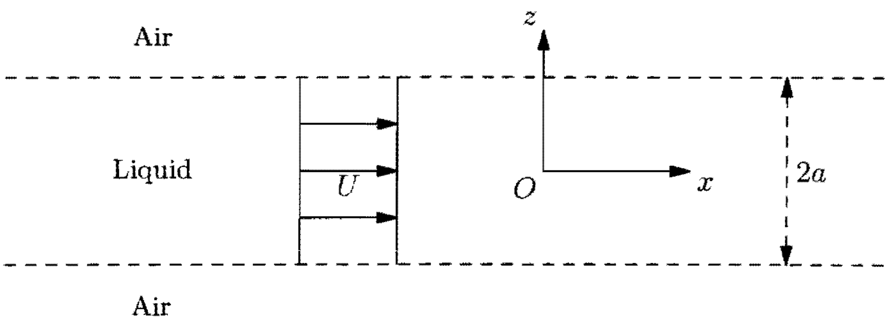

Fig. 1

2 (a) Consider a fluid between two horizontal plates a distance d apart. The lower and upper plates are maintained at steady uniform temperatures of T_0 and T_1 respectively.

(i) By considering the motion of a fluid particle, show that the equilibrium configuration of no fluid motion is unstable to inviscid perturbations if $T_0 > T_1$. [20%]

(ii) What is the non-dimensional number that determines the stability of the liquid? What does it mean physically? [20%]

(b) The following linear coupled ordinary differential equations for a fluid blob, with vertical velocity $w(t)$ and temperature $\theta(t)$, represent a simplified model of Rayleigh-Bénard instability:

$$\begin{aligned}\frac{\mathrm{d}w(t)}{\mathrm{d}t} &= -\frac{\nu}{d^2}w(t) + \alpha g(\theta(t) - \bar{\theta}) \\ \frac{\mathrm{d}\theta(t)}{\mathrm{d}t} &= -\frac{\kappa}{d^2}(\theta(t) - \bar{\theta}) + \frac{T_0 - T_1}{d}w(t)\end{aligned}$$

where ν, κ, α, and g are positive constants, and $\bar{\theta}$ is a constant temperature.

(i) Show that $\theta = \bar{\theta}$ and $w = 0$ corresponds to an equilibrium state. [10%]

(ii) Show that this equilibrium state is unstable when

$$\alpha g \frac{(T_0 - T_1)}{d} \frac{d^3}{\nu\kappa} > 1$$

[50%]

3 In flow instability, the evolution of a small perturbation, $u(x,t)$, is sometimes modelled by the one-dimensional advection-diffusion equation

$$\frac{\partial u}{\partial t} + a_1 \frac{\partial u}{\partial x} = a_2 \frac{\partial^2 u}{\partial x^2}$$

where x is the spatial coordinate and t is time.

(a) In this context, what do the parameters a_1 and a_2 represent physically? [10%]

(b) Assume that the perturbation takes the form $u(x,t) = u_0 e^{i(kx - \omega t)}$, where k is the wavenumber and ω is the angular frequency.

(i) Derive an expression for the phase velocity, ω/k. [10%]

(ii) Derive an expression for the group velocity, $d\omega/dk$. [10%]

(iii) Determine whether this system is dispersive or non-dispersive. [10%]

(c) Consider the case in which $a_1 = -1 + \alpha i$ and $a_2 = 1$ and express your results in terms of α, which is a real number.

(i) Find the range of wavenumbers over which the perturbation is temporally unstable and sketch the temporal stability curve, $\omega_i(k_r)$. [20%]

(ii) Remembering that k can be complex, find the value of k at which the group velocity is zero. [10%]

(iii) Determine the values of α over which this flow is absolutely unstable. [30%]

4 A bluff body with mass m and height D is held by a spring with stiffness k in a fluid with density ρ and speed U. The damping factor, ζ, is small and the undamped natural frequency of the system is ω. The vertical displacement of the body, y, satisfies

$$m\ddot{y} + p\dot{y} + ky = qU^2 c_y$$

where $p = 2m\zeta\omega$ and $q = \rho D/2$. The force coefficient of the bluff body, c_y, is related to the angle of attack, α, by $c_y = \alpha - r\alpha^3$, where $r > 0$. In the following questions, you may leave your answers in terms of p and q.

(a) Explain why $\alpha \approx \dot{y}/U$ when $\dot{y} \ll U$ and express c_y in terms of U and $\dot{y}$. [10%]

(b) Write down an expression for the total energy, E, of the mass–spring system in terms of m, $\dot{y}$, k, and y. Hence show that

$$\frac{\mathrm{d}E}{\mathrm{d}t} = (qU - p)\dot{y}^2 - \frac{qr}{U}\dot{y}^4$$

Comment on how the terms on the right hand side vary over one period of oscillation. [30%]

(c) Assume that a periodic solution exists of the form $y = a\cos\omega t$. By integrating $\mathrm{d}E/\mathrm{d}t$ over one period, show that the system first becomes unstable when $qU > p$. Explain your reasoning clearly. You may assume without proof that

$$\int_0^{2\pi/\omega} \sin^2\omega t \,\mathrm{d}t = \pi/\omega,$$

$$\int_0^{2\pi/\omega} \sin^4\omega t \,\mathrm{d}t = 3\pi/(4\omega).$$

[30%]

(d) When $qU > p$, oscillations grow until the amplitude, a, no longer changes in time. Using the result from part (c), derive an expression for this amplitude in terms of the other variables. Sketch a graph of a as a function of U and comment on this dependence. [30%]

END OF PAPER

4A10, 2012, Answers

Q1 –

Q2 –

Q3

(a) –

(b) $\omega/_k = a_1 - \mathrm{i}a_2 k$; $d\omega/_{dk} = a_1 - 2\mathrm{i}a_2 k$

(c) $k = (\alpha + \mathrm{i})/2$; $\alpha^2 > 1$

Q4

(a) –

(b) –

(c) –

(d) $a^2 = \frac{4}{3qr\omega^2} U(qU - p)$