ENGINEERING TRIPOS PART IIB

Wednesday 9 May 2012 2:30 to 4

Module 4A12

TURBULENCE AND VORTEX DYNAMICS

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachments:
4A12 Data Card: (i) Vortex Dynamics (1 page);
(ii) Turbulence (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

PAD03

1 (a) Sketch the secondary flow pattern in a Bödewadt layer. Explain the physical origin of the secondary flow. [20%]

(b) Consider a steady, laminar Bödewadt layer on an infinite disk. We wish to estimate the boundary-layer thickness, δ, and the magnitude of the axial flow just outside the layer, $|u_z|_{\text{out}}$, in terms of the viscosity, ν, and fluid angular velocity, Ω.

(i) Use order-of-magnitude analysis to show that, within the Bödewadt layer, the radial and azimuthal velocities, u_r and u_θ, are related by

$$u_\theta^2/r \sim \nu u_r/\delta^2$$

(ii) Taking $u_r \sim u_\theta$ within the boundary layer, show that the boundary-layer thickness is $\delta \sim \sqrt{\nu/\Omega}$.

(iii) Now use continuity to show that $|u_z|_{\text{out}} \sim \sqrt{\nu\Omega}$.

(iv) Show that the estimates $\delta \sim \sqrt{\nu/\Omega}$ and $|u_z|_{\text{out}} \sim \sqrt{\nu\Omega}$ also follow directly from dimensional analysis. [40%]

(c) A cylindrical teacup of radius R is partially filled with tea. The tea is set in rotation with angular velocity Ω and the free surface is more or less flat. We wish to determine the core flow *outside* the Bödewadt and side-wall boundary layers as the fluid slowly spins down. We use cylindrical polar coordinates (r, θ, z), where $z = 0$ is close to the base of the cup and located at the top of the Bödewadt layer, and $z = H$ is the free surface. The core flow outside the boundary layers is axisymmetric and takes the form $\left(u_r^{(\text{core})}, \Omega r, u_z^{(\text{core})}\right)$.

(i) Taking $u_r \sim u_\theta$ within the Bödewadt layer, use continuity to show that

$$u_r^{(\text{core})} H \sim \delta \Omega r \sim \sqrt{\nu\Omega}\, r$$

and hence confirm that

$$u_z^{(\text{core})} \sim \sqrt{\nu\Omega}\left[1 - z/H\right]$$

(ii) Estimate the spin-down time for the flow in terms of ν, Ω and H. [40%]

2 (a) A two-dimensional vortex has an azimuthal velocity distribution in cylindrical polar coordinates, (r, θ, z), given by

$$u_\theta = \frac{\Gamma_0}{2\pi r}\left[1 - \exp\left(-r^2/\delta^2\right)\right]$$

where Γ_0 is the net circulation surrounding the vortex and δ is its characteristic radius. Confirm that the associated vorticity points in the z direction and has the distribution

$$\omega = \frac{\Gamma_0}{\pi \delta^2}\exp\left[-r^2/\delta^2\right]$$

Show that this represents an exact solution of the two-dimensional vorticity equation

$$\frac{D\omega}{Dt} = \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \omega}{\partial r}\right)$$

provided that $\delta = c\sqrt{\nu t}$ for some constant c. Find c. [60%]

(b) The vortex is now subject to the external strain field $u_r = -\frac{1}{2}\alpha r$, $u_z = \alpha z$, where α is a strain-rate. For the particular case of $\alpha = 4\nu/\delta^2$ we have Burgers vortex, and the flow is steady with $\delta = \text{constant}$. Describe the different physical processes which balance in Burgers vortex to give a steady flow. What would you expect to happen to the vortex if $\alpha > 4\nu/\delta^2$ or else $\alpha < 4\nu/\delta^2$? Why has Burgers vortex attracted considerable attention in the turbulence community in recent years? [40%]

3 Consider the ideal situation of stationary homogeneous isotropic non-decaying turbulence with integral lengthscale L and kinetic energy k.

(a) Discuss briefly the concept of the *energy cascade* and provide the rationale behind the usual estimate for the rate of dissipation of energy ε. [20%]

(b) The fluid is heated so that its kinematic viscosity changes from ν to 2ν, with L and k remaining the same. How do the rate of dissipation and the Taylor lengthscale change? Also, how do the Kolmogorov length, time and velocity scales change? [20%]

(c) With reference to the turbulent kinetic energy equation, discuss whether the concept of isotropic homogeneous non-decaying turbulence is physically realisable. [20%]

(d) In a more detailed view of the structure of isotropic turbulence, it is assumed that all the dissipation occurs in small-scale vortical structures of thickness equal to the Kolmogorov lengthscale, across which the velocity is assumed to vary by an amount equal to $\sqrt{k}$. Estimate the rate of dissipation in the small-scale structures and hence provide an order-of-magnitude estimate for the volume fraction of small-scale structures in the fluid. [40%]

4 Consider the turbulent mixing layer formed downstream of a splitter plate separating a stream with velocity $U_1 = 50\,\mathrm{ms}^{-1}$ from a stream with velocity $U_2 = 10\,\mathrm{ms}^{-1}$.

(a) Sketch carefully the distributions of the mean streamwise velocity and the three normal Reynolds stresses across the mixing layer, including a discussion about the relative magnitude of the latter. [25%]

(b) At a particular distance from the splitter plate, the width of the layer is 50 mm. By making reasonable estimates for the turbulence lengthscale and velocity scale at the middle of the layer, estimate the integral timescale of the velocity signal as seen by a stationary hot-wire probe at that location. [25%]

(c) Assuming complete self-preservation and starting from the thin-shear flow equations

$$\frac{\partial \overline{u}}{\partial x} + \frac{\partial \overline{v}}{\partial y} = 0$$

$$\overline{u}\frac{\partial \overline{u}}{\partial x} + \overline{v}\frac{\partial \overline{u}}{\partial y} = -\frac{\partial \overline{u'v'}}{\partial y}$$

show that the characteristic width of the layer δ is proportional to x, the streamwise distance measured from the start of the shear layer, and derive the governing equations for the self-similar velocity functions. State clearly the assumptions and closures made. [50%]

END OF PAPER

ENGINEERING TRIPOS PART IIB 2012

MODULE 4A12: TURBULENCE AND VORTEX DYNAMICS.

NUMERICAL ANSWERS

Q1. –
Q2. (a) $c = 2$, (b) -
Q3. -
Q4. (a) - , (b) ~ 0.33 ms (c) -