ENGINEERING TRIPOS PART IIB

Wednesday 25 April 2012 2.30 to 4

Module 4F3

OPTIMAL AND PREDICTIVE CONTROL

*Answer not more than **three** questions.*

All questions carry the same number of marks.

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

Attachment: Data sheet (2 pages).

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed.

You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator

kg04

1 (a) A discrete-time system satisfies the state equation,

$$x_{k+1} = f(x_k, u_k)$$

with x_0 given. It is desired to determine $u_0, u_1, \cdots, u_{h-1}$ to minimise the cost,

$$J(x_0, u_0, u_1, \cdots, u_{h-1}) = \sum_{k=0}^{h-1} c(x_k, u_k) + J_h(x_h)$$

where $u_k \in U$ for all k.

(i) Derive the dynamic programming equation for the 'cost to go' or value function, $V(x, k)$. [20%]

(ii) Discuss the power and limitations of this approach. [10%]

(b) A gambler wins a large sum, S, and decides to cease working and gambling and live off this sum for the remainder of his life, which he knows to be T years. He also knows that the rate of return for this investment will be α, so that his remaining capital, $x(t)$, will satisfy the differential equation,

$$\frac{dx(t)}{dt} = \alpha x(t) - u(t)$$

where $u(t)$ is his spend rate. He wishes to maximise his total utility which is given by

$$\int_0^T \sqrt{u(t)}\, dt$$

(i) Write down the Hamilton-Jacobi-Bellman equation for this problem, with boundary conditions respecting the gambler's wish to leave nothing on his death. Hence determine the partial differential equation to be satisfied by the value function. [25%]

(ii) Show that the value function given by

$$V(x, t) = \sqrt{x} \times \sqrt{w(t)}$$

will satisfy the H-J-B equation for $w(t)$ satisfying a differential equation. [25%]

(iii) Hence calculate the optimal spend rate $u(t)$ as a function of $x(t)$. [20%]

2 A system is described by the state equation,

$$\begin{aligned}\dot{x}(t) &= Ax(t) + B_1 w(t) + B_2 u(t)\\ z(t) &= C_1 x(t) + D_{12} u(t)\end{aligned}$$

where u is the control input, w is a disturbance input and the state is measured and available for feedback control. It is desired to determine a control law such that $\|T_{w\to z}\|_\infty < \gamma$. An algebraically convenient assumption is to make

$$\begin{bmatrix} C_1 & D_{12} \end{bmatrix} = \begin{bmatrix} C_{11} & 0 \\ 0 & I \end{bmatrix} \tag{1}$$

However this is **not** the case here and the change of variables,

$$\hat{u} = Ru + Lx, \quad \hat{z} = Mz$$

is to be considered to transform the equations so that equation (1) is satisfied for the new variables (R, L and M are appropriately dimensioned constant matrices with R and M square and invertible).

(a) Find $\hat{A}$, $\hat{B}_1$, $\hat{B}_2$, $\hat{C}_1$ and $\hat{D}_{12}$ such that,

$$\dot{x}(t) = \hat{A}x(t) + \hat{B}_1 w(t) + \hat{B}_2 \hat{u}(t), \quad \hat{z}(t) = \hat{C}_1 x(t) + \hat{D}_{12}\hat{u}(t)$$

[10%]

(b) Give a sufficient condition on M to ensure that $\|T_{w\to z}\|_\infty < \gamma$ is equivalent to $\|T_{w\to \hat{z}}\|_\infty < \gamma$. [20%]

(c) Assume that the singular value decomposition of D_{12} is given by,

$$D_{12} = [U_1 \ U_2] \begin{bmatrix} \Sigma \\ 0 \end{bmatrix} V^T$$

where $U = [U_1 \ U_2]$ and V are square matrices satisfying $U^T U = I$ and $V^T V = I$ and $\det \Sigma \neq 0$. Determine the appropriate values for R, L and M for $\hat{D}_{12}$ and $\hat{C}_1$ to satisfy equation (1). [30%]

(d) Determine the appropriate algebraic Riccati equation to solve this revised problem, and hence determine the control law for the original problem in terms of the solution, X, of this algebraic Riccati equation. [40%]

3 (a) The standard form of predictive control employs a receding horizon. Explain the principle of the receding horizon and suggest, with an example, why predictive control might be attractive in an industrial setting. [15%]

(b) A linear plant with state x_k at time k and input u_k at time k is described by a discrete-time state space model:

$$x_{k+1} = Ax_k + Bu_k.$$

Letting x_0 be the measured state at the current time, show that a sequence of the current and predicted future states $\mathbf{x} \triangleq [x_0^T, x_1^T, x_2^T, x_3^T]^T$ can be expressed in terms of x_0 and a sequence of inputs $\mathbf{u} \triangleq [u_0^T, u_1^T, u_2^T]^T$ in the form $\mathbf{x} = \Phi x_0 + \Gamma \mathbf{u}$. [20%]

(c) For compatibly sized matrices Q, R, S and P, a particular control design for the system in part (b) minimises the finite horizon cost function

$$V(\mathbf{x}, \mathbf{u}) = x_3^T P x_3 + \sum_{k=0}^{2} \left(x_k^T Q x_k + u_k^T R u_k + x_k^T S u_k + u_k^T S^T x_k \right).$$

(i) Give conditions on matrices P, Q, R and S that will ensure the optimisation problem is convex. [5%]

(ii) Some constraints on the plant states and inputs have been specified by the inequalities $J\mathbf{u} - Wx_0 \leq c$, and you are also asked to include the terminal constraint $x_3 = 0$ as a means of guaranteeing closed loop stability. Letting

$$\mathcal{Q}_e = \begin{bmatrix} Q & 0 & 0 & 0 \\ 0 & Q & 0 & 0 \\ 0 & 0 & Q & 0 \\ 0 & 0 & 0 & P \end{bmatrix}, \quad \mathcal{R}_e = \begin{bmatrix} R & 0 & 0 \\ 0 & R & 0 \\ 0 & 0 & R \end{bmatrix}, \quad \mathcal{S}_e = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & S \\ 0 & 0 & 0 \end{bmatrix}$$

express the minimisation of $V(\mathbf{x}, \mathbf{u})$ subject to these constraints in the standard form of a quadratic program (QP), with decision variable $\mathbf{u}$ in terms of Φ, Γ, $\mathcal{Q}_e$, $\mathcal{R}_e$, $\mathcal{S}_e$, J, c, W and x_0. [40%]

(d) What are the computational implications of implementing predictive control and how might these be addressed? [20%]

4 A particular single-state linear plant is perfectly described by the discrete-time model $x_{k+1} = 1.2x_k + u_k$. To avoid damage to the actuators, the input must remain within the constraints $|u_k| \leq 5$.

(a) (i) Considering the above system in closed loop with static feedback gain $u_k = Kx_k$, explain the term *constraint admissible set.* [10%]

(ii) Find the stabilising feedback gain K such that the *constraint admissible set* for $u_k = Kx_k$ is equal to $-3 \leq x_k \leq 3$ and show that this set is *invariant* for the controlled system. [20%]

(b) For the above system and positive scalars q, r and p, a given predictive controller minimises the finite horizon cost function

$$V(\mathbf{x}, \mathbf{u}) = px_N^2 + \sum_{k=0}^{N-1} \left(qx_k^2 + ru_k^2\right)$$

subject to the constraints $|u_k| \leq 5$ for $k \in \{0, \ldots, N-1\}$ and $|x_N| \leq 3$.

(i) An optimal input sequence $\mathbf{u}^* = [u_0^*, u_1^*, \ldots, u_{N-1}^*]^T$ with a corresponding predicted state sequence $\mathbf{x}^* = [x_0, x_1^*, \ldots, x_N^*]^T$ is found, with value function $V^*(x_0) = V(\mathbf{x}^*, \mathbf{u}^*)$. The control action u_0^* is applied to the plant. A new optimisation is performed for the new plant state. Show that the value function of the new problem satisfies:

$$V^*(x_1) \leq V^*(x_0) - qx_0^2 - ru_0^{*2} - px_N^{*\,2} + (q + K^2 r)x_N^{*\,2} + (1.2 + K)^2 px_N^{*\,2}.$$

where K was determined in part (a)(ii) [40%]

(ii) Write down an inequality in terms of p, K, q and r that ensures $V^*(x_k)$ is a *Control Lyapunov Function.* What conclusion can you draw when this inequality is satisfied? [20%]

(c) For what values of x_k does a stabilising controller <u>not</u> exist? If values of x_k in this region are expected during normal plant operation, what should you do? [10%]

END OF PAPER