ENGINEERING TRIPOS PART IIB

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Thursday 3 May 2012 2:30 to 4

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Module 4F5

ADVANCED WIRELESS COMMUNICATIONS

*Answer not more than **three** questions.*

*All questions carry the same number of marks.*

*The **approximate** percentage of marks allocated to each part of a question is indicated in the right margin.*

*There are no attachments.*

STATIONERY REQUIREMENTS
Single-sided script paper

SPECIAL REQUIREMENTS
Engineering Data Book
CUED approved calculator allowed

**You may not start to read the questions printed on the subsequent pages of this question paper until instructed that you may do so by the Invigilator**

js04

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1 Three signals $x_1(.)$, $x_2(.)$ and $x_3(.)$ are defined for $t \in [-1,1]$ as

$$x_1(t) = t, \quad x_2(t) = t^2, \text{ and } \quad x_3(t) = t^3$$

and $x_k(t) = 0$ for $t \notin [-1,1]$ for $k = 1,2,3$.

(a) Compute the norm of each signal. [15%]

(b) Compute the inner products $\langle x_1(.),\, x_2(.)\rangle$, $\langle x_1(.),\, x_3(.)\rangle$ and $\langle x_2(.),\, x_3(.)\rangle$. [10%]

(c) Using the Gram-Schmidt procedure, construct an orthonormal basis for the signal space spanned by $x_1(.)$, $x_2(.)$ and $x_3(.)$. [25%]

(d) What are the coordinates of the vector representations $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ of $x_1(.)$, $x_2(.)$ and $x_3(.)$ respectively in the orthonormal basis constructed in part (c)? [15%]

(e) Compute the norms of the vectors $\mathbf{x}_1$, $\mathbf{x}_2$ and $\mathbf{x}_3$ and verify that they are equal to the norms of the corresponding signals, and compute the distance between $\mathbf{x}_1$ and $\mathbf{x}_2$, between $\mathbf{x}_2$ and $\mathbf{x}_3$, and between $\mathbf{x}_1$ and $\mathbf{x}_3$. [20%]

(f) Assuming equiprobable messages, an Additive White Gaussian Noise (AWGN) channel with power spectral density $N_0/2$ W/Hz and a cross-correlation demodulator, use the union bound to express an upper bound on the uncoded error probability for optimal detection. [15%]

*Hint:* the result of part (f) can be expressed using the $Q$ function, which you are not expected to evaluate numerically.

2 Figure 1 represents a Binary Erasure Channel (BEC) with binary input random variable $X$ and ternary output random variable $Y$. The diagram shows the channel transition probabilities $P_{Y|X}(y|x)$ for all values $x$ and $y$ of $X$ and $Y$, respectively.

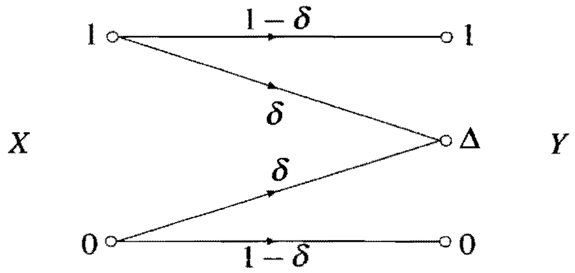


Fig. 1

The output random variable $Y$ of the BEC is forwarded to a processor that applies a function $f(.)$ to yield the binary random variable $Z = f(Y)$, as shown in Fig. 2.

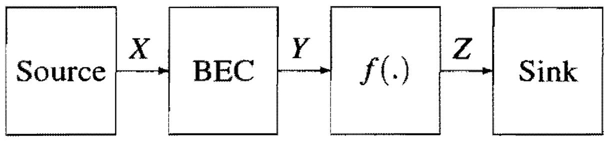


Fig. 2

(a) Explain why the conditional entropy $H(X|Y)$ is equal to $H(X|Y,Z)$. Can the mutual information $I(X;Z)$ exceed the mutual information $I(X;Y)$? Justify your answer. [15%]

(b) Let $f(0) = f(1) = 0$ and $f(\Delta) = 1$. Such a function $f(.)$ is often used in information theory and is called an "indicator function" because it indicates whether the channel output is erased or not. Supposing that $P_X(1) = 1 - P_X(0) = p$, compute the probability distribution of $Z$ and show that it doesn't depend on $p$. Give values for $H(X|Y,Z=0)$ and for $H(X|Y,Z=1)$ and justify these values. [15%]
*Hint:* you may use the binary entropy function $H_2(p) = -p\log p - (1-p)\log(1-p)$.

(c) Express the mutual information $I(X;Y)$ as a function of $p$. What is the value of $p$ that maximises mutual information? Justify your answer. Explain why the quantity you have just computed is the capacity of the BEC. [15%]

(d) For $\delta = 0.6$, we want to transmit information over the BEC using an encoder of rate $R = 1/2$. Can the probability of a decoding error be made arbitrarily small, i.e., smaller than any given $\varepsilon$, under these constraints? If yes, describe an approach for achieving arbitrary reliability. If not, state a strictly positive lower bound on the error probability as a function of $R$, of the capacity $C$, and of the codeword length $n$. [20%]

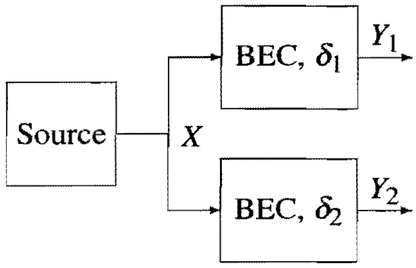


Fig. 3

(e) Now consider two independent BECs in parallel with parameters $\delta_1$ and $\delta_2$, the same input $X$, and outputs $Y_1$ and $Y_2$, as shown in Fig. 3. Using an indicator function as you did in the previous parts, express the mutual information $I(X;Y_1,Y_2)$ of the combined channel and hence the capacity of the channel $X \longrightarrow Y_1,Y_2$. [20%]

*Hint:* the binary-valued indicator function should be of the form $Z = f(Y_1,Y_2)$.

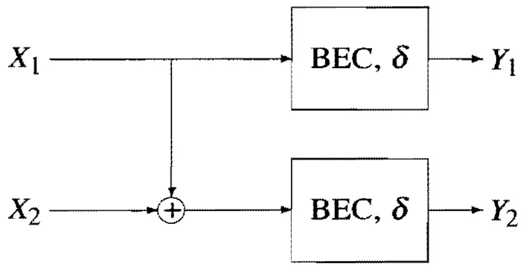


Fig. 4

(f) The construction represented in Fig. 4 is a building block in a new coding technique called "polar coding". The inputs $X_1$ and $X_2$ are binary, independent, and $P_{X_1}(0) = P_{X_1}(1) = P_{X_2}(0) = P_{X_2}(1) = 1/2$. The sum is modulo 2. Decoding operates as "successive decoding", in which knowledge of previously decoded inputs is assumed when decoding subsequent inputs. Specifically, the decoder for the building block depicted first decodes $X_2$, then decodes $X_1$ knowing $X_2$. The relevant channels are thus

**Channel 1:** $X_1 \longrightarrow Y_1,Y_2,X_2$

**Channel 2:** $X_2 \longrightarrow Y_1,Y_2$

Compute the mutual informations $I(X_1;Y_1,Y_2,X_2)$ and $I(X_2;Y_1,Y_2)$. [15%]

3 Figure 5 represents a convolutional encoder where for each use of the encoder, $b_k$ is an input binary digit at time $k$, $c_{1,k}$ and $c_{2,k}$ are the corresponding binary code digits at time $k$, and $s_{1,k}$ and $s_{2,k}$ are the binary state variables, i.e., $s_{1,k} = b_{k-1}$ and $s_{2,k} = b_{k-2}$. The sum operators are modulo 2.

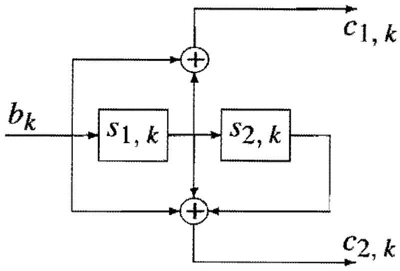


Fig. 5

(a) State the rate of this convolutional encoder and express its generators in octal form. [20%]

(b) How many states does the encoder have? Draw the state diagram of the encoder, labeling each state transition with the corresponding input and two output digits. [20%]

(c) Assume that the encoder starts at time $k = 0$ with state initiated to $s_{1,0} = s_{2,0} = 0$. Draw the terminated trellis diagram for this encoder for an information sequence of length $L = 4$ followed by an appropriate number of zero-valued termination digits to ensure that the encoder returns to its initial state. [30%]

*Hint:* Leave sufficient space on the margins of your drawing as you may want to re-use this trellis in part (e).

(d) The output of the decoder is transmitted via the discrete memoryless channel described in the transition diagram in Fig. 6. State the maximum-likelihood decoding rule and use logarithms to turn it into an additive path metric maximisation rule. Compute a corresponding metric table for the channel in Fig. 6, adding a constant to ensure that all metrics are non-negative. In what way does the added constant affect the maximum-likelihood decoding rule? [10%]

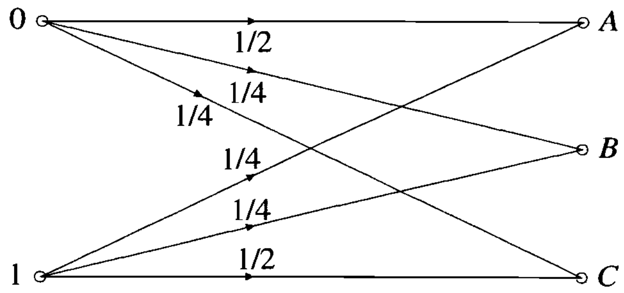


Fig. 6

(e) The received sequence at the output of the channel is

$$C, C, B, A, C, A, B, C, A, B, B, A.$$

Using the Viterbi algorithm, find a maximum likelihood information sequence for the convolutional encoder in Fig. 5 and the channel in Fig. 6. [20%]

*Hint:* you may work out the maximum likelihood decoding path directly on the trellis you drew in part (c) and do not need to re-draw a trellis for this part.

4 (a) You want to design an indoor Wireless Local Area Network (WLAN) system that operates at carrier frequency 5 GHz, over a bandwidth of 20 MHz, and uses codewords of duration 3.5 $\mu$s. You may assume that the mobile users are moving at a velocity of $v$=10 km/h. You may further assume that the delay spread for an indoor environment is 100 ns and that Jakes' model for the Doppler power spectrum applies, i.e.,

$$S_H(\xi) = \begin{cases} \frac{1}{\pi f_m} \frac{1}{\sqrt{1-(\xi/f_m)^2}}, & |\xi| < f_m \\ 0, & |\xi| \geq f_m \end{cases}$$

where $f_m$ is the maximum Doppler frequency. The speed of light can be approximated as $c = 3 \times 10^8 \mathrm{m/s}$.

(i) Is the considered fading channel underspread or overspread? Justify your answer. [20%]

(ii) Is the channel selective in time? Is it selective in frequency? Justify your answer. [10%]

(b) Consider the frequency-selective fading channel

$$y_i = \sum_{\ell=0}^{N_p-1} h_\ell \, x_{i-\ell} + n_i, \quad i = 1, 2, \ldots$$

where the channel taps $h_0, \ldots, h_{N_p-1}$ are independent, have zero mean and the same variance $\sigma^2$.

(i) Explain in words why the independence of $h_0, \ldots, h_{N_p-1}$ is a consequence of the Uncorrelated Scatterers (US) assumption. [10%]

(ii) We wish to transmit data over this channel using Orthogonal Frequency-Division Multiplexing (OFDM). Draw a block-diagram of an OFDM system and explain the basic idea behind OFDM. [10%]

(iii) Show that for the Discrete Fourier Transform (DFT) coefficients

$$H_i = \sum_{\ell=0}^{N_p-1} h_\ell \, e^{-j\frac{2\pi i \ell}{n}}, \quad i = 0, \ldots, n-1$$

to be uncorrelated, $N_p$ must be an integer multiple of $n$.

[20%]

*Hint:* You may find the identity $\sum_{\ell=0}^{N-1} \alpha^\ell = \frac{1-\alpha^N}{1-\alpha}$ useful.

(iv) Assume that $N_p = 7$. Consider the linear code whose generator matrix is given by

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{pmatrix}$$

What is the diversity achieved by the code when transmitted using OFDM using a DFT of the same length as the code? [20%]

(v) Does this code achieve the same diversity when $N_p = 2$? Justify your answer. [10%]

**END OF PAPER**

# 4F5 2012 Exam Answers

(1) (a) $\|x_1(.)\| = \frac{\sqrt{6}}{3}$, $\|x_2(.)\| = \frac{\sqrt{10}}{5}$, $\|x_3(.)\| = \frac{\sqrt{14}}{7}$

(b) $\langle x_1(.), x_2(.)\rangle = 0$, $\langle x_1(.), x_3(.)\rangle = \frac{2}{5}$, $\langle x_2(.), x_3(.)\rangle = 0$.

(c) For $t \in [-1, 1]$, $f_1(t) = \frac{\sqrt{6}}{2}t$, $f_2(t) = \frac{\sqrt{10}}{2}t^2$, and $f_3(t) = \frac{5\sqrt{14}}{4}\left(t^3 - \frac{3}{5}t\right)$.

(d) $\boldsymbol{x}_1 = (\frac{\sqrt{6}}{3}, 0, 0)$, $\boldsymbol{x}_2 = (0, \frac{\sqrt{10}}{5}, 0)$, $\boldsymbol{x}_3 = (\frac{\sqrt{6}}{5}, 0, \frac{2\sqrt{14}}{35})$,

(e) $\|\boldsymbol{x}_1 - \boldsymbol{x}_2\| = \frac{4\sqrt{15}}{15}$, $\|\boldsymbol{x}_2 - \boldsymbol{x}_3\| = \frac{2\sqrt{210}}{35}$, $\|\boldsymbol{x}_3 - \boldsymbol{x}_1\| = \frac{4\sqrt{105}}{105}$.

(f) $P_e \leq \frac{2}{3}\left[Q\left(\frac{4\sqrt{15}}{15}\right) + Q\left(\frac{2\sqrt{210}}{35}\right) + Q\left(\frac{4\sqrt{105}}{105}\right)\right]$

(2) (a) ...

(b) $P_Z(1) = \delta$, $H(X|Y, Z = 0) = 0$, $H(X|Y, Z = 1) = H_2(p)$,

(c) $I(X;Y) = H_2(p)(1-\delta)$, $\max_p I(X;Y) = 1 - \delta$.

(d) $P_e \geq \frac{1}{5} - \frac{2}{n}$.

(e) $I(X;Y_1,Y_2) = H_2(p)(1 - \delta_1\delta_2)$, $C = \max_{P_X} I(X;Y_1,Y_2) = 1 - \delta_1\delta_2$.

(f) $I(X_2;Y_1,Y_2) = (1-\delta)^2$, $I(X_1;X_2,Y_1,Y_2) = 1 - \delta^2$.

(3) (a) $R = 1/2$, generator polynomials $(6,7)_8$.

(b) 4 states

(c) ...

(d)

| $x_i \backslash y_i$ | A | B | C |
|---|---|---|---|
| 0 | 1 | 0 | 0 |
| 1 | 0 | 0 | 1 |

(e) 1,1,0,0

(4) (a) (i) channel underspread

(ii) flat in time, selective in frequency

(b) (i) ...

(ii) ...

(iii) ...

(iv) diversity 3

(v) diversity less than 3.